

H¹-CONFORMING APPROXIMATION OF THE MAXWELL EQUATIONS IN HETEROGENEOUS MEDIA WITH MINIMAL REGULARITY

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ABSTRACT. The present paper proposes and analyzes a nodal C^0 -finite element technique to solve the Maxwell equations in domains with heterogeneous properties. The convergence analysis is done assuming only minimal regularity in Lipschitz domains. The method is shown to converge for any polynomial degrees and is spectrally correct.

1. INTRODUCTION

The objectives of the present paper is to propose and analyze a nodal C^0 -finite element technique to solve the Maxwell equations in domains with heterogeneous properties. More precisely, given a three-dimensional domain Ω with boundary Γ , we want to construct an approximation of the following problem using C^0 -Lagrange finite elements:

$$(1.1) \quad \nabla \times (\kappa \nabla \times \mathbf{E}) = \varepsilon \mathbf{g}, \quad \nabla \cdot (\varepsilon \mathbf{E}) = 0, \quad \mathbf{E} \times \mathbf{n}|_{\Gamma} = 0.$$

This task is non-trivial, since it is known since the pioneering work of Costabel [11] that \mathbf{H}^1 -conforming approximation techniques that rely on uniform L^2 -stability estimates both on the curl and the divergence of the approximate field do not converge properly if Ω is non-smooth and non-convex. This defect is a consequence of $\mathbf{H}^1(\Omega) \cap \mathbf{H}_{0,\text{curl}}(\Omega)$ being a closed proper subspace of $\mathbf{H}_{\text{div}}(\Omega) \cap \mathbf{H}_{0,\text{curl}}(\Omega)$. This is probably one reason why edge elements have been favored over C^0 -Lagrange finite elements over the years. It is only recently, say since the ground-breaking “rehabilitation” work of Costabel and Dauge [12], Bramble and Pasciak [6] and Bramble et al. [7] that C^0 -Lagrange finite elements have regained their status as credible approximation tools for the Maxwell equations and more generally for div-curl problems. The key idea developed in the above references is that the divergence of the discrete field approximating $\varepsilon \mathbf{E}$ must be controlled in a space that is intermediate between $L^2(\Omega)$ and $H^{-1}(\Omega)$. This program is carried out in Costabel and Dauge [12] by controlling the divergence of $\varepsilon \mathbf{E}$ in a weighted L^2 -space

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where the weight is a distance to the re-entrant corners of the domain to some appropriate power depending on the strength of the singularity. The analysis of the method by Costabel and Dauge [12] requires the approximation space to contain the gradient of C^1 scalar-valued functions, which excludes low-order finite-elements spaces. This restriction is removed in Buffa et al. [10] by considering a mixed form of the weighted L^2 -stabilization technique. The method developed by Bramble and Pasciak [6] and Bramble et al. [7] involves a least-square approximation of a discrete problem with different test and trial spaces. The trial space is $\mathbf{L}^2(\Omega)$ and the components of the test space are subspaces of $H^1(\Omega)$. The numerical method uses piecewise constant functions for the trial space and piecewise linear functions enriched with face bubbles for the test space. A technique based on a local L^2 -stabilization of the divergence of $\varepsilon \mathbf{E}$ and using finite elements of order high enough so as to contain the gradient of Argyris or Hsieh-Clough-Tocher C^1 -finite elements is introduced in Duan et al. [15]. The convergence analysis of the method requires the source term to be smooth enough so that $\nabla \times \mathbf{E} \in \mathbf{H}^r(\Omega)$ with $r > \frac{1}{2}$. This method is further revisited in two space dimensions in Duan et al. [14] to allow for low-order finite elements and to remove the smoothness assumption on $\nabla \times \mathbf{E}$.

The present paper is the second part of a research program started in Bonito and Guermond [4]. The technique adopted in [4] consists of stabilizing the divergence of the field $\varepsilon \mathbf{E}$ in a negative Sobolev norm through a mixed formulation. It is shown in [4] that stabilizing the divergence in $H^{-1}(\Omega)$ is sufficient to solve the boundary value problem (1.1), but it may not be sufficient in general to solve the associated eigenvalue problem if only Lipschitz regularity of the domain is assumed. In that case the divergence must be stabilized in $H^{-\alpha}(\Omega)$ with $\alpha \in (\frac{\ell}{2\ell-1}, 1]$ where $\ell - 1$ is the polynomial degree of the approximation of \mathbf{E} , $\ell \geq 1$. Note in passing that the method introduced in [4] with the particular choice $\alpha = 1$ has also been proposed in Badia and Codina [3]. The convergence analysis of the boundary value problem therein assumes that the right-hand is divergence free and either the solution to (1.1) is smooth or the degree of the finite element space is large enough so as to contain the gradient of C^1 scalar-valued functions. The method proposed in Bonito and Guermond [4] converges for all $\alpha \in (\frac{\ell}{2\ell-1}, 1]$ as stated in [4, Lemma 5.4], and the convergence rate is even maximal when $\alpha = 1$ *without* the extra assumptions used in [3], provided the right-hand of the boundary value problem solenoidal (which is usually the case). Yet, the possibility of choosing $\alpha < 1$ has been introduced in [4] to ensure the spectral correctness of the approximation for eigenvalue problems.

The objective of the present paper is to generalize the analysis of Bonito and Guermond [4] to boundary and eigenvalue problems with coefficients κ and ε in (1.1) that are piecewise smooth. Our analysis assumes only the natural regularity of the solution; in particular the a priori regularity of \mathbf{E} may be lower than that of $\mathbf{H}^{\frac{1}{2}}(\Omega)$, see Theorem 2.1. The approximation is done using a mixed formulation with nodal finite elements and an interior penalty method to account for the jumps in the coefficients κ and ε . There is no restriction on the polynomial degree of the approximation in the convergence analysis (besides that it is continuous). One essential argument of this paper is the construction of a smoothing operator in $\mathbf{H}_{0,\text{curl}}(\Omega)$ that commutes (almost) with the curl operator, see Lemma 3.6. In passing we correct a mistake from [4] where the smoothing operator was not constructed properly. The second important argument is Lemma A.3 in the Appendix. This is a variant of Lemma 8.2 in Buffa and Perugia [9]; however, our proof slightly

differs from that in Buffa and Perugia [9] since the estimates therein do not seem to be uniform in the meshsize.

The paper is organized as follows. We introduce notation and recall a priori regularity results in §2. The smoothing operator in $\mathbf{H}_{0,\text{curl}}(\Omega)$ is introduced in §3. The key properties of this operator are stated in Theorem 3.1 and Lemma 3.6 (the estimate (3.17) is particularly important). The finite element framework and the interior penalty technique are presented in §4. This section also contains stability estimates for the weak formulation of the boundary value problem. The convergence analysis for the boundary value problem is done in §5. The two important results from this section are Theorem 5.1 and Theorem 5.3. Theorem 5.1 establishes convergence in a discrete norm and Theorem 5.3 establishes convergence in $\mathbf{L}^2(\Omega)$ using a duality argument. The spectral correctness of the approximation of the eigenvalue problem is analyzed in §6. The paper is complemented with an appendix containing technical details. Lemma A.3 is one of the key results from the Appendix.

2. PRELIMINARIES

2.1. Spaces. Let D be an open connected Lipschitz domain in \mathbb{R}^3 . (In the rest of the paper D denotes a generic open Lipschitz domain that may differ from Ω .) The space of smooth functions with compact support in D is denoted $\mathcal{D}(D)$. The norm in $H^1(D)$ is defined as follows:

$$(2.1) \quad \|v\|_{H^1(D)}^2 := \|v\|_{L^2(D)}^2 + \|\nabla v\|_{\mathbf{L}^2(D)}^2.$$

The space $H^s(D)$ for $s \in (0, 1)$ is defined by the method of real interpolation between $H^1(D)$ and $L^2(D)$ (see e.g. Tartar [22]), i.e.,

$$(2.2) \quad H^s(D) = [L^2(D), H^1(D)]_{s,2}.$$

We also define the space $H_0^1(D)$ to be the completion of $\mathcal{D}(D)$ with respect to the following norm:

$$(2.3) \quad \|v\|_{H_0^1(D)} := \|\nabla v\|_{\mathbf{L}^2(D)}.$$

This allows us again to define the space $H_0^s(D)$ for $s \in (0, 1)$ by the method of real interpolation between $H_0^1(D)$ and $L^2(D)$ as follows:

$$(2.4) \quad H_0^s(D) = [L^2(D), H_0^1(D)]_{s,2}.$$

(This definition is slightly different from what is usually done; the only differences occurs at $s = \frac{1}{2}$. What we hereafter denote by $H_0^{\frac{1}{2}}(D)$ is usually denoted by $H_{00}^{\frac{1}{2}}(D)$ elsewhere.) Recall that the spaces $H_0^s(D)$ and $H^s(D)$ coincide for $s \in [0, \frac{1}{2})$ and their norms are equivalent, (see e.g. Lions and Magenes [19, Thm 11.1], Girault and Raviart [16, Cor 1.4.4.5], or Tartar [22, Chap. 33]). The space $H^{-s}(D)$ is defined by duality with $H_0^s(D)$ for $0 \leq s \leq 1$, i.e., with a slight abuse of notation we define

$$\|v\|_{H^{-s}(D)} = \sup_{0 \neq w \in H_0^s(D)} \frac{\int_D vw}{\|w\|_{H_0^s(D)}}.$$

It is a standard result that $H^{-s}(D) = [L^2(D), H^{-1}(D)]_{s,2}$.

The above definitions are naturally extended to the vector-valued Sobolev spaces $\mathbf{H}^s(D)$ and $\mathbf{H}_0^s(D)$. We additionally introduce the following spaces of vector-valued

functions:

$$(2.5) \quad \mathbf{H}_{\text{curl}}(D) = \{\mathbf{v} \in \mathbf{L}^2(D) \mid \nabla \times \mathbf{v} \in \mathbf{L}^2(D)\},$$

$$(2.6) \quad \mathbf{H}_{0,\text{curl}}(D) = \{\mathbf{v} \in \mathbf{L}^2(D) \mid \nabla \times \mathbf{v} \in \mathbf{L}^2(D), \mathbf{v} \times \mathbf{n}|_{\partial D} = 0\},$$

$$(2.7) \quad \mathbf{H}_{\text{curl}}^r(D) = \{\mathbf{v} \in \mathbf{L}^2(D) \mid \nabla \times \mathbf{v} \in \mathbf{H}^r(D)\},$$

$$(2.8) \quad \mathbf{H}_{0,\text{curl}}^r(D) = \{\mathbf{v} \in \mathbf{L}^2(D) \mid \nabla \times \mathbf{v} \in \mathbf{H}^r(D), \mathbf{v} \times \mathbf{n}|_{\partial D} = 0\},$$

all equipped with their natural norm; for instance, $\|\mathbf{v}\|_{\mathbf{H}_{\text{curl}}(D)}^2 = \|\mathbf{v}\|_{\mathbf{L}^2(D)}^2 + \|\nabla \times \mathbf{v}\|_{\mathbf{L}^2(D)}^2$.

2.2. The domain. The domain Ω is a bounded open set in \mathbb{R}^3 . The boundary of Ω , say Γ , is assumed to have the Lipschitz regularity and to be connected. To simplify the presentation we also assume that $0 \in \Omega$ and Ω is star-shaped with respect to an open neighborhood of 0. More precisely, there is an open neighborhood of the origin, say \mathcal{V} , such that Ω is star-shaped with respect to all the points on \mathcal{V} . This assumption is equivalent to assuming that Ω is a star-shaped domain with respect to the origin and there exists a real number $\chi \in (0, 1)$ such that the following holds for any $\delta \in (0, 1)$:

$$(2.9) \quad (1 - \delta)\Omega + B(0, \delta\chi) \subset\subset \Omega,$$

where $B(0, r)$ is the ball centered at 0 of radius r and $\subset\subset$ denotes compact embedding.

A key piece of the convergence analysis of the method that we propose in this paper is based on the existence of a family of smoothing operators in $\mathbf{H}_{0,\text{curl}}(\Omega)$. This construction is discussed in detail in §3. The main challenge one encounters when constructing this family of operators is to make it compatible with the boundary condition and to commute with the curl operator. The purpose of the hypothesis (2.9) is to make this construction possible. The hypothesis (2.9) may seem restrictive, but, using a partition of unity technique, we expect the results presented in this paper to remain valid for any domain that can be divided into finitely many domains that are star-shaped with respect to a open neighborhood.

2.3. Mixed formulation of the problem. It will prove convenient to reformulate the original problem (1.1) in mixed form to have a better control on the divergence of the field $\varepsilon \mathbf{E}$. More precisely, from now on we consider the following problem: Given a vector field \mathbf{g} , find \mathbf{E} and p such that

$$(2.10) \quad \nabla \times (\kappa \nabla \times \mathbf{E}) + \varepsilon \nabla p = \varepsilon \mathbf{g}; \quad \nabla \cdot (\varepsilon \mathbf{E}) = 0, \quad \mathbf{E} \times \mathbf{n}|_{\Gamma} = 0, \quad p|_{\Gamma} = 0.$$

The two problems (2.10) and (1.1) are equivalent if $\nabla \cdot (\varepsilon \mathbf{g}) = 0$, since in this case $p = 0$ in (2.10) (apply the divergence operator to the first equation).

The scalar fields κ and ε are assumed to be piecewise smooth. More precisely we assume that Ω is partitioned into N Lipschitz subdomains $\Omega_1, \dots, \Omega_N$ such that the restrictions of κ and ε to these subdomains are smooth. To better formalize this assumption we define

$$(2.11) \quad \Sigma := \bigcup_{i \neq j} \partial\Omega_i \cap \partial\Omega_j,$$

$$(2.12) \quad W_{\Sigma}^{1,\infty}(\Omega) := \{\nu \in L^{\infty}(\Omega) \mid \nabla(\nu|_{\Omega_i}) \in \mathbf{L}^{\infty}(\Omega_i), i = 1, \dots, N\}.$$

We refer to Σ as the interface between the subdomains Ω_i . In the rest of the paper we assume that the fields ε and κ satisfy the following properties: There exist ε_{\min} , $\kappa_{\min} > 0$ such that

$$(2.13) \quad \varepsilon, \kappa \in W_{\Sigma}^{1,\infty}(\Omega), \quad \text{and} \quad \varepsilon \geq \varepsilon_{\min}, \quad \kappa \geq \kappa_{\min} \quad \text{a.e. in } \Omega.$$

The following stability results proved in Bonito et al. [5] play important roles in the stability of the finite element method developed in this paper:

Theorem 2.1. *Assuming that $\varepsilon \mathbf{g} \in \mathbf{L}^2(\Omega)$ and (2.13), Problem (2.10) has a unique solution in $\mathbf{H}_{0,\text{curl}}(\Omega)$. Moreover, there exist c , τ_{ε} and τ_{κ} , depending on Ω and the fields ε and κ , so that*

$$(2.14) \quad \|\mathbf{E}\|_{\mathbf{H}^s(\Omega)} \leq c \|\varepsilon \mathbf{g}\|_{\mathbf{L}^2(\Omega)}, \quad \forall s \in [0, \tau_{\varepsilon}),$$

$$(2.15) \quad \|\nabla \times \mathbf{E}\|_{\mathbf{H}^s(\Omega)} \leq c \|\varepsilon \mathbf{g}\|_{\mathbf{L}^2(\Omega)}, \quad \forall s \in [0, \tau_{\kappa}).$$

$$(2.16) \quad \|\nabla \times (\kappa \nabla \times \mathbf{E})\|_{\mathbf{L}^2(\Omega)} + \|\nabla p\|_{L^2(\Omega)} \leq c \|\varepsilon \mathbf{g}\|_{\mathbf{L}^2(\Omega)},$$

Remark 2.1. *In general the regularity indices τ_{ε} and τ_{κ} are smaller than $\frac{1}{2}$ when the domain Ω is not convex and the scalar field ε and κ are discontinuous across Σ .*

3. SMOOTH APPROXIMATION IN $\mathbf{H}_{0,\text{curl}}(\Omega)$

We introduce in this section a smoothing operator in $\mathbf{H}_{0,\text{curl}}(\Omega)$ that will be used to prove the convergence of the finite element approximation. The key difficulty that we are facing is to find an approximation that is compatible with the boundary condition in $\mathbf{H}_{0,\text{curl}}(\Omega)$ and commutes with the curl operator. We essentially proceed as in Bonito and Guermond [4] but modify the argument to correct an incorrect statement made therein. When invoking $\mathcal{C}_h(\mathbf{Ae})_{\varepsilon}$ in the proof of Lemma 3.3 in Bonito and Guermond [4] it was incorrectly assumed that $(\mathbf{Ae})_{\varepsilon}$ is in $\mathbf{H}_{0,\text{curl}}(\Omega)$, which is not the case. We resolve this issue in the present construction by introducing an additional scaling operator, S_D^{δ} .

3.1. Extension operator. Let D be an open Lipschitz domain in \mathbb{R}^3 . For any $\mathbf{F} \in \mathbf{L}^1(D)$, we denote $E_D \mathbf{F}$ the extension of \mathbf{F} by 0, i.e.,

$$(3.1) \quad E_D \mathbf{F}(\mathbf{x}) = \begin{cases} \mathbf{F}(\mathbf{x}) & \text{if } \mathbf{x} \in D, \\ 0 & \text{elsewhere.} \end{cases}$$

Let $\delta \in [0, \frac{1}{2}]$, define $\bar{\delta} := 1 - \delta$ and set $D_{\delta} := \bar{\delta}D$. We define the scaling operator $S_D^{\delta} : \mathbf{L}^1(D) \mapsto \mathbf{L}^1(D_{\delta})$ by

$$(3.2) \quad \forall \mathbf{F} \in \mathbf{L}^1(D), \quad \forall \mathbf{x} \in D_{\delta}, \quad (S_D^{\delta} \mathbf{F})(\mathbf{x}) := \mathbf{F}(\mathbf{x} \bar{\delta}^{-1}).$$

Lemma 3.1. *The following commuting properties hold:*

$$(3.3) \quad S_{\mathbb{R}^3}^{\delta} E_D = E_{D_{\delta}} S_D^{\delta}$$

$$(3.4) \quad \partial_{x_i} (S_D^{\delta} \mathbf{F}) = \bar{\delta}^{-1} S_D^{\delta} (\partial_{x_i} \mathbf{F}), \quad \forall \mathbf{F} \in \mathbf{L}^1(D), \quad \forall i = 1, \dots, d,$$

$$(3.5) \quad \nabla \times (E_D \mathbf{F}) = E_D (\nabla \times \mathbf{F}), \quad \forall \mathbf{F} \in \mathbf{H}_{0,\text{curl}}(D),$$

$$(3.6) \quad \nabla (E_D \mathbf{F}) = E_D (\nabla \mathbf{F}), \quad \forall \mathbf{F} \in \mathbf{H}_0^1(D).$$

Proof. (3.3) is evident and (3.4) is just the chain rule. We only prove (3.5) since the proof of the (3.6) is similar. Let \mathbf{F} be a member of $\mathbf{H}_{0,\text{curl}}(D)$, then the following holds:

$$\langle \nabla \times (E_D \mathbf{F}), \phi \rangle = \int_{\mathbb{R}^3} (E_D \mathbf{F}) \cdot \nabla \times \phi = \int_D \mathbf{F} \cdot \nabla \times \phi = \int_D \nabla \times \mathbf{F} \cdot \phi, \quad \forall \phi \in \mathcal{D}(\mathbb{R}^3),$$

where the last equality holds owing to \mathbf{F} being in $\mathbf{H}_{0,\text{curl}}(D)$. Then

$$\langle \nabla \times (E_D \mathbf{F}), \phi \rangle = \int_{\mathbb{R}^3} E_D (\nabla \times \mathbf{F}) \cdot \phi, \quad \forall \phi \in \mathcal{D}(\mathbb{R}^3),$$

which proves the statement. \square

Lemma 3.2. *For all $r \in [0, 1]$, (i) the linear operator $E_D : \mathbf{H}_0^r(D) \rightarrow \mathbf{H}_0^r(\mathbb{R}^3)$ is bounded; (ii) the family of operators $\{S_D^\delta\} : \mathbf{H}^r(D) \rightarrow \mathbf{H}^r(D_\delta)$ is uniformly bounded with respect to $\delta \in [0, \frac{1}{2}]$.*

Proof. It is a standard result that E_D is a continuous operator from $\mathbf{L}^2(D)$ to $\mathbf{L}^2(\mathbb{R}^3)$ and from $\mathbf{H}_0^1(D)$ to $\mathbf{H}_0^1(\mathbb{R}^3)$, Adams and Fournier [1]. Then the first assertion follows directly from the interpolation theory. For the second part, a scaling argument ensures that S_D^δ is a continuous operator from $\mathbf{L}^2(D)$ to $\mathbf{L}^2(D_\delta)$. Using (3.4), we infer that it is also a continuous operator from $\mathbf{H}^1(D)$ to $\mathbf{H}^1(D_\delta)$. The conclusion follows from the interpolation theory. \square

Taking $r \in [0, \frac{1}{2})$ and using the fact that the spaces $\mathbf{H}_0^r(\Omega)$ and $\mathbf{H}^r(\Omega)$ coincide (with equivalent norms), we infer that there exists c such that,

$$(3.7) \quad \forall \mathbf{F} \in \mathbf{H}^r(\Omega), \quad \|E_\Omega \mathbf{F}\|_{\mathbf{H}^r(\mathbb{R}^3)} \leq c \|\mathbf{F}\|_{\mathbf{H}^r(\Omega)}.$$

Moreover, using this inequality and the second part of Lemma 3.2 with $D = \mathbb{R}^3$, we infer that $S_{\mathbb{R}^3}^\delta E_\Omega : \mathbf{H}^r(\Omega) \rightarrow \mathbf{H}^r(\mathbb{R}^3)$ is a linear continuous operator, and there exists c , uniform in δ , such that

$$(3.8) \quad \forall \mathbf{F} \in \mathbf{H}^r(\Omega), \quad \|S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{H}^r(\mathbb{R}^3)} \leq c \|\mathbf{F}\|_{\mathbf{H}^r(\Omega)}.$$

Lemma 3.3. *The following holds:*

$$(3.9) \quad \forall \mathbf{F} \in \mathbf{H}_{0,\text{curl}}(\Omega), \quad \nabla \times (S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}) = \bar{\delta}^{-1} S_{\mathbb{R}^3}^\delta E_\Omega (\nabla \times \mathbf{F}).$$

Proof. Let $\mathbf{F} \in \mathbf{H}_{0,\text{curl}}(\Omega)$. By (3.4) we infer that

$$\nabla \times (S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}) = \bar{\delta}^{-1} S_{\mathbb{R}^3}^\delta \nabla \times (E_\Omega \mathbf{F}).$$

Then (3.5) from Lemma 3.1 implies

$$\nabla \times (S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}) = \bar{\delta}^{-1} S_{\mathbb{R}^3}^\delta E_\Omega (\nabla \times \mathbf{F}),$$

since $\mathbf{F} \in \mathbf{H}_{0,\text{curl}}(\Omega)$. This concludes the proof. \square

Lemma 3.4. *The linear operator $S_{\mathbb{R}^3}^\delta E_\Omega : \mathbf{H}_{0,\text{curl}}^r(\Omega) \rightarrow \mathbf{H}_{0,\text{curl}}^r(\mathbb{R}^3)$ is bounded for all $r \in [0, \frac{1}{2})$. More precisely there is c , uniform with respect to δ , so that the following holds:*

$$(3.10) \quad \|\nabla \times (S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F})\|_{\mathbf{H}^r(\mathbb{R}^3)} \leq c \|\nabla \times \mathbf{F}\|_{\mathbf{H}^r(\Omega)}.$$

Proof. The identity (3.9) implies that $S_{\mathbb{R}^3}^\delta E_\Omega$ is a continuous map from $\mathbf{H}_{0,\text{curl}}(\Omega)$ to $\mathbf{H}_{0,\text{curl}}(\mathbb{R}^3)$. Let $r \in [0, \frac{1}{2})$ and let \mathbf{F} be a member of $\mathbf{H}_{0,\text{curl}}^r(\Omega)$. A simple scaling argument implies that $S_\Omega^\delta \mathbf{F}$ is a member of $\mathbf{H}_{0,\text{curl}}^r(\Omega_\delta)$. Since $\nabla \times S_\Omega^\delta \mathbf{F}$ is in $\mathbf{H}^r(\Omega)$ and $r \in [0, \frac{1}{2})$, the extension by zero is stable in $\mathbf{H}^r(\mathbb{R}^3)$, i.e., $E_{\Omega_\delta} \nabla \times S_\Omega^\delta \mathbf{F}$ is a member of $\mathbf{H}^r(\mathbb{R}^3)$ and there is a constant c , uniform with respect to \mathbf{F} and δ , so that

$$\begin{aligned} \|E_{\Omega_\delta} \nabla \times S_\Omega^\delta \mathbf{F}\|_{\mathbf{H}^r(\mathbb{R}^3)} &\leq c' \|\nabla \times S_\Omega^\delta \mathbf{F}\|_{\mathbf{H}^r(\Omega_\delta)} = c' \bar{\delta}^{-1} \|S_\Omega^\delta \nabla \times \mathbf{F}\|_{\mathbf{H}^r(\Omega_\delta)} \\ &\leq c \|\nabla \times \mathbf{F}\|_{\mathbf{H}^r(\Omega)}. \end{aligned}$$

Note that c is uniform with respect to δ since $\bar{\delta} \in [\frac{1}{2}, 1]$. Then, applying (3.3) and (3.5) to the above inequality gives

$$\begin{aligned} \|\nabla \times (S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F})\|_{\mathbf{H}^r(\mathbb{R}^3)} &= \|\nabla \times (E_{\Omega_\delta} S_\Omega^\delta \mathbf{F})\|_{\mathbf{H}^r(\mathbb{R}^3)} = \|E_{\Omega_\delta} \nabla \times S_\Omega^\delta \mathbf{F}\|_{\mathbf{H}^r(\mathbb{R}^3)} \\ &\leq c \|\nabla \times \mathbf{F}\|_{\mathbf{H}^r(\Omega)}, \end{aligned}$$

which concludes the proof. \square

We now state a lemma that gives some important approximation properties of the operator $\mathbf{F} \mapsto S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}$.

Lemma 3.5. *There exists K_1 for all $r \in [0, 1]$ such that the following holds for every $\mathbf{F} \in \mathbf{H}_0^r(\Omega)$:*

$$(3.11) \quad \|\mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{H}_0^s(\Omega)} \leq K_1 \delta^{r-s} \|\mathbf{F}\|_{\mathbf{H}_0^r(\Omega)} \quad 0 \leq s \leq r \leq 1,$$

and for all $r \in [0, \frac{1}{2})$ there exists $K_2(r)$, such that the following holds every $\mathbf{F} \in \mathbf{H}_{0,\text{curl}}^r(\Omega)$

$$(3.12) \quad \|\nabla \times (\mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F})\|_{\mathbf{H}^s(\Omega)} \leq K_2(r) \delta^{r-s} \|\nabla \times \mathbf{F}\|_{\mathbf{H}^r(\Omega)} \quad 0 \leq s \leq r < \frac{1}{2}.$$

Proof. We prove the first inequality (3.11) by means of an interpolation technique. Using Lemma 3.1 together with $d \geq 2$, we have

$$\begin{aligned} \|\mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{L}^2(\Omega)} &\leq \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)} + \|S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{L}^2(\Omega)} \leq \left(1 + \bar{\delta}^{\frac{d}{2}}\right) \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)} \leq 2 \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}. \\ \|\mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{H}_0^1(\Omega)} &= \|\nabla (\mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F})\|_{\mathbf{L}^2(\Omega)} \leq \|\nabla \mathbf{F}\|_{\mathbf{L}^2(\Omega)} + \|\nabla S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{L}^2(\Omega)} \\ &= \|\nabla \mathbf{F}\|_{\mathbf{L}^2(\Omega)} + \bar{\delta}^{-1} \|S_{\mathbb{R}^3}^\delta \nabla (E_\Omega \mathbf{F})\|_{\mathbf{L}^2(\Omega)} = \|\nabla \mathbf{F}\|_{\mathbf{L}^2(\Omega)} + \bar{\delta}^{\frac{d}{2}-1} \|E_\Omega \nabla \mathbf{F}\|_{\mathbf{L}^2(\Omega)} \\ &= \left(1 + \bar{\delta}^{\frac{d}{2}-1}\right) \|\nabla \mathbf{F}\|_{\mathbf{L}^2(\Omega)} \leq 2 \|\mathbf{F}\|_{\mathbf{H}_0^1(\Omega)}. \end{aligned}$$

We now derive an estimate for the mapping $\mathbf{H}_0^1(\Omega) \ni \mathbf{F} \mapsto \mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F} \in \mathbf{L}^2(\Omega)$. The definition of $S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}$ implies that

$$\begin{aligned} \|\mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{L}^2(\Omega)}^2 &= \int_\Omega |(E_\Omega \mathbf{F})(\mathbf{x}) - (E_\Omega \mathbf{F})(\mathbf{x} \bar{\delta}^{-1})|^2 \, d\mathbf{x} \\ &= \int_\Omega \left| \int_0^1 \nabla (E_\Omega \mathbf{F})((1-t)\mathbf{x} + t\mathbf{x} \bar{\delta}^{-1}) \cdot \frac{\delta}{\bar{\delta}} \mathbf{x} \, dt \right|^2 \, d\mathbf{x} \\ &\leq \int_\Omega \frac{\delta^2}{\bar{\delta}^2} |\mathbf{x}|^2 \int_0^1 |\nabla (E_\Omega \mathbf{F})((1-t)\mathbf{x} + t\mathbf{x} \bar{\delta}^{-1})|^2 \, dt \, d\mathbf{x} \end{aligned}$$

Then, we introduce $M := \max_{\mathbf{x} \in \Omega} |\mathbf{x}|$, and we apply Fubini's lemma:

$$\|\mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{L}^2(\Omega)}^2 \leq M^2 \frac{\delta^2}{\bar{\delta}^2} \int_0^1 \int_\Omega |\nabla(E_\Omega \mathbf{F})((1-t)\mathbf{x} + t\mathbf{x}\bar{\delta}^{-1})|^2 d\mathbf{x} dt$$

Using a change of variable in the inner integral, we finally obtain

$$\begin{aligned} \|\mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{L}^2(\Omega)}^2 &\leq M^2 \frac{\delta^2}{\bar{\delta}^2} \|\nabla E_\Omega \mathbf{F}\|_{\mathbf{L}^2(\Omega_\delta)}^2 \int_0^1 \left(\frac{\bar{\delta}}{\bar{\delta} + \delta t} \right)^d dt \\ &\leq M^2 \delta^2 \bar{\delta}^{-2} \|\nabla(E_\Omega \mathbf{F})\|_{\mathbf{L}^2(\mathbb{R}^3)}^2. \end{aligned}$$

Since $\mathbf{F} \in \mathbf{H}_0^1(\Omega)$, we have $\|\nabla E_\Omega \mathbf{F}\|_{\mathbf{L}^2(\mathbb{R}^3)} = \|E_\Omega \nabla \mathbf{F}\|_{\mathbf{L}^2(\mathbb{R}^3)} = \|\nabla \mathbf{F}\|_{\mathbf{L}^2(\Omega)}$. Using now the assumption $\delta \leq \frac{1}{2}$, i.e., $\bar{\delta}^{-1} \leq 2$, we finally deduce that

$$(3.13) \quad \|\mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{L}^2(\Omega)} \leq 2M\delta \|\nabla \mathbf{F}\|_{\mathbf{L}^2(\Omega)} = 2M\delta \|\mathbf{F}\|_{\mathbf{H}_0^1(\Omega)}.$$

We now set $K_1 := \max(2, 2M)$ and we have proven that

$$\begin{aligned} \|\mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{L}^2(\Omega)} &\leq K_1 \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}, \\ \|\mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{L}^2(\Omega)} &\leq K_1 \delta \|\mathbf{F}\|_{\mathbf{H}_0^1(\Omega)}, \\ \|\mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{H}_0^1(\Omega)} &\leq K_1 \|\mathbf{F}\|_{\mathbf{H}_0^1(\Omega)}. \end{aligned}$$

We conclude that (3.11) holds by using the Lions-Peetre Reiteration Theorem.

We now turn our attention to (3.12). Let $r \in [0, \frac{1}{2})$ and consider $s \in [0, r]$. Let \mathbf{F} be a member of $\mathbf{H}_{0,\text{curl}}^r(\Omega)$. We observe first that $S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}$ is also in $\mathbf{H}_{0,\text{curl}}^r(\Omega)$ owing to Lemma 3.4. Using (3.9) gives

$$\begin{aligned} \|\nabla \times (\mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F})\|_{\mathbf{H}_0^s(\Omega)} &= \|\nabla \times \mathbf{F} - \bar{\delta}^{-1} S_{\mathbb{R}^3}^\delta E_\Omega \nabla \times \mathbf{F}\|_{\mathbf{H}_0^s(\Omega)} \\ &\leq \|\nabla \times \mathbf{F} - \bar{\delta}^{-1} \nabla \times \mathbf{F}\|_{\mathbf{H}_0^s(\Omega)} + \bar{\delta}^{-1} \|\nabla \times \mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega (\nabla \times \mathbf{F})\|_{\mathbf{H}_0^s(\Omega)} \\ &\leq \delta \bar{\delta}^{-1} \|\nabla \times \mathbf{F}\|_{\mathbf{H}_0^s(\Omega)} + K_1 \bar{\delta}^{-1} \delta^{r-s} \|\nabla \times \mathbf{F}\|_{\mathbf{H}_0^r(\Omega)}. \end{aligned}$$

Using $\delta < \frac{1}{2}$, i.e., $\bar{\delta}^{-1} \leq 2$, we have

$$\|\nabla \times (\mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F})\|_{\mathbf{H}_0^s(\Omega)} \leq 2(K_1 + \delta^{1-r+s}) \delta^{r-s} \|\nabla \times \mathbf{F}\|_{\mathbf{H}_0^r(\Omega)},$$

Remembering that $\mathbf{H}^s(\Omega)$ and $\mathbf{H}_0^s(\Omega)$ coincide for $0 \leq r < \frac{1}{2}$ and that their norm are equivalent, the above inequality yields (3.12) since $1 - r + s \geq 1 - r > 0$. \square

3.2. Smooth approximation. We now use the above extension operator $S_{\mathbb{R}^3}^\delta E_\Omega$ together with a mollification to construct a smooth approximation operator. For $\delta \in (0, \frac{1}{2})$, we set

$$(3.14) \quad \rho_\delta(\mathbf{x}) := \delta^{-d} \rho(\mathbf{x}/\delta), \text{ where } \rho(\mathbf{x}) := \begin{cases} \eta \exp\left(-\frac{1}{1-|\mathbf{x}|^2}\right), & \text{if } |\mathbf{x}| < 1, \\ 0, & \text{if } |\mathbf{x}| \geq 1, \end{cases}$$

where η is chosen so that $\int_{\mathbb{R}^d} \rho = 1$. We define a family of approximation operators $\{\mathcal{K}_\delta\}_{\delta>0}$ in the following way:

$$(3.15) \quad \mathcal{K}_\delta \mathbf{F} = \rho_{\delta\chi} \star (S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}), \quad \forall \mathbf{F} \in \mathbf{L}^1(\Omega)$$

where χ is the constant introduced in (2.9).

Theorem 3.1. $\mathcal{K}_\delta \mathbf{F}|_\Omega$ is in $\mathcal{C}_0^\infty(\Omega)$ for all $\mathbf{F} \in \mathbf{L}^1(\Omega)$. There exists a constant K such that the following estimates hold for any $0 < \delta < \frac{1}{2}$:

$$(3.16) \quad \|\mathbf{F} - \mathcal{K}_\delta \mathbf{F}\|_{\mathbf{H}_0^s(\Omega)} \leq K \delta^{r-s} \|\mathbf{F}\|_{\mathbf{H}_0^r(\Omega)} \quad 0 \leq s \leq r \leq 1$$

$$(3.17) \quad \|\nabla \times \mathbf{F} - \nabla \times \mathcal{K}_\delta \mathbf{F}\|_{\mathbf{H}^s(\Omega)} \leq K \delta^{r-s} \|\nabla \times \mathbf{F}\|_{\mathbf{H}^r(\Omega)} \quad 0 \leq s \leq r < \frac{1}{2}$$

$$(3.18) \quad \|\mathcal{K}_\delta \mathbf{F}\|_{\mathbf{H}^r(\Omega)} \leq K \chi^{s-r} \delta^{s-r} \|\mathbf{F}\|_{\mathbf{H}^s(\Omega)} \quad 0 \leq s \leq r, s < \frac{1}{2}$$

and all $\mathbf{F} \in \mathbf{H}_0^r(\Omega)$, all $\mathbf{F} \in \mathbf{H}_{0,\text{curl}}(\Omega)$, and all $\mathbf{F} \in \mathbf{H}^r(\Omega)$, respectively.

Proof. Owing to the properties of the mollification operator, we have $\mathcal{K}_\delta \mathbf{F}|_\Omega \in \mathcal{C}^\infty(\Omega)$. We now prove that the support of $\mathcal{K}_\delta \mathbf{F}$ is compact in Ω . The definition of the convolution operation implies that the following holds for all $\mathbf{x} \in \mathbb{R}^d$:

$$\mathcal{K}_\delta \mathbf{F}(\mathbf{x}) = \int_{\mathbb{R}^d} (S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F})(\mathbf{y}) \rho_{\delta\chi}(\mathbf{x} - \mathbf{y}) \, d\mathbf{y} = \int_{\bar{\delta}\Omega} \mathbf{F}(\mathbf{y}/\bar{\delta}) \rho_{\delta\chi}(\mathbf{x} - \mathbf{y}) \, d\mathbf{y}.$$

If $\mathbf{x} \notin \bar{\delta}\Omega + B(0, \delta\chi)$, then for all $\mathbf{y} \in \bar{\delta}\Omega$, we have $\rho_{\delta\chi}(\mathbf{x} - \mathbf{y}) = 0$ and then $\mathcal{K}_\delta \mathbf{F}(\mathbf{x}) = 0$. As a result, $\mathcal{K}_\delta \mathbf{F}$ is supported in $\bar{\delta}\Omega + B(0, \delta\chi)$ which is compactly embedded in Ω owing to the assumption (2.9). Hence, $\mathcal{K}_\delta \mathbf{F} \in \mathcal{C}_0^\infty(\Omega)$; in particular, we have $\mathcal{K}_\delta \mathbf{F} \in \mathbf{H}_{0,\text{curl}}(\Omega)$. We now prove the estimates (3.16) to (3.18). Let us first consider $\mathbf{F} \in \mathbf{H}_0^r(\Omega)$. Using standard approximation properties of the mollification operator, we deduce that there exists a uniform constant $K_3 > 0$ so that

$$\|S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F} - \mathcal{K}_\delta \mathbf{F}\|_{\mathbf{H}_0^s(\Omega)} \leq K_3 (\delta\chi)^{r-s} \|S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{H}_0^r(\mathbb{R}^3)}, \quad 0 \leq s \leq r \leq 1.$$

Using the triangle inequality and Lemma 3.5 we have

$$\begin{aligned} \|\mathbf{F} - \mathcal{K}_\delta \mathbf{F}\|_{\mathbf{H}_0^s(\Omega)} &\leq \|\mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{H}_0^s(\Omega)} + \|S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F} - \mathcal{K}_\delta \mathbf{F}\|_{\mathbf{H}_0^s(\Omega)} \\ &\leq K_1 \delta^{r-s} \|\mathbf{F}\|_{\mathbf{H}_0^r(\Omega)} + K_3 \chi^{r-s} \delta^{r-s} \|S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{H}_0^r(\mathbb{R}^3)} \\ &\leq (K_1 + 2K_3 \chi^{r-s}) \delta^{r-s} \|\mathbf{F}\|_{\mathbf{H}_0^r(\Omega)}. \end{aligned}$$

This proves (3.16) with $K = K_1 + 2K_3$ since $\chi \leq 1$ and $s \leq r$. Let us now consider $\mathbf{F} \in \mathbf{H}_{0,\text{curl}}^r(\Omega)$. Using that $\nabla \times \mathcal{K}_\delta \mathbf{F} = \rho_{\delta\chi} \star \nabla \times (S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F})$, we infer that

$$\|\nabla \times (S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F} - \mathcal{K}_\delta \mathbf{F})\|_{\mathbf{H}^s(\Omega)} \leq K_3 (\delta\chi)^{r-s} \|\nabla \times (S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F})\|_{\mathbf{H}^r(\mathbb{R}^3)} \quad 0 \leq s \leq r$$

Using the triangle inequality together with (3.10), Lemma 3.5, and assuming that $r < \frac{1}{2}$ we have

$$\begin{aligned} \|\nabla \times (\mathbf{F} - \mathcal{K}_\delta \mathbf{F})\|_{\mathbf{H}^s(\Omega)} &\leq \|\nabla \times (\mathbf{F} - S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F})\|_{\mathbf{H}^s(\Omega)} + \|\nabla \times (S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F} - \mathcal{K}_\delta \mathbf{F})\|_{\mathbf{H}^s(\Omega)} \\ &\leq K_2(r) \delta^{r-s} \|\nabla \times \mathbf{F}\|_{\mathbf{H}^r(\Omega)} + K_3 (\delta\chi)^{r-s} \|\nabla \times (S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F})\|_{\mathbf{H}^r(\mathbb{R}^3)} \\ &\leq (K_2(r) + K_2 \chi^{r-s}) \|\nabla \times \mathbf{F}\|_{\mathbf{H}^r(\Omega)}, \end{aligned}$$

which proves (3.17) with $K = K_1 + 2K_2(r)$ since $\chi \leq 1$ and $s \leq r$. Let us finally assume that $\mathbf{F} \in \mathbf{H}^r(\Omega)$. Using again the properties of the mollification operator, we infer that

$$\|\mathcal{K}_\delta \mathbf{F}\|_{\mathbf{H}^r(\Omega)} \leq \|\mathcal{K}_\delta \mathbf{F}\|_{\mathbf{H}^r(\mathbb{R}^3)} \leq K_3 (\delta\chi)^{s-r} \|S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F}\|_{\mathbf{H}^s(\mathbb{R}^3)} \quad 0 \leq s \leq r.$$

Applying (3.8), we obtain (3.18). Note that the assumption $s < \frac{1}{2}$ is required in order to ensure that $S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F} \in \mathbf{H}^r(\mathbb{R}^3)$. \square

Remark 3.1. In the following, we will use (3.18) without mentioning χ^{s-r} in the right hand sides. Indeed, we will use the inequality with r bounded from above by the polynomial degree of the approximation; as a result, χ^{s-r} is uniformly bounded.

We end this section by mentioning a key commuting property on \mathcal{K}_δ .

Lemma 3.6. *The following holds for any $\mathbf{F} \in \mathbf{H}_{\text{curl}}(\Omega)$:*

$$(3.19) \quad \bar{\delta} \nabla \times \mathcal{K}_\delta \mathbf{F} = \mathcal{K}_\delta (\nabla \times \mathbf{F}).$$

Proof. Owing to the properties of the convolution, the following holds for any $\mathbf{F} \in \mathbf{H}_{\text{curl}}(\Omega)$:

$$\nabla \times \mathcal{K}_\delta \mathbf{F} = \rho_{\delta\chi} \star (\nabla \times (S_{\mathbb{R}^3}^\delta E_\Omega \mathbf{F})).$$

Applying (3.9), we infer that

$$\begin{aligned} \nabla \times \mathcal{K}_\delta \mathbf{F} &= \rho_{\delta\chi} \star (\bar{\delta}^{-1} S_{\mathbb{R}^3}^\delta E_\Omega (\nabla \times \mathbf{F})) \\ &= \bar{\delta}^{-1} \rho_{\delta\chi} \star (S_{\mathbb{R}^3}^\delta E_\Omega (\nabla \times \mathbf{F})) = \bar{\delta}^{-1} \mathcal{K}_\delta (\nabla \times \mathbf{F}). \end{aligned}$$

This completes the proof. \square

4. FINITE ELEMENT APPROXIMATION OF THE BOUNDARY VALUE PROBLEM

We introduce and study the stability properties of a Lagrange finite element technique for solving the boundary value problem (2.10).

4.1. Finite Element Spaces. Let $\{\mathcal{T}_h\}_{h>0}$ be a shape regular family of conforming affine meshes with typical mesh size h . We assume that the sub-domains Ω_i , $i = 1, \dots, N$ are polyhedra and the interface Σ is captured by the meshes in $\{\mathcal{T}_h\}_{h>0}$, i.e., Σ is partitioned by a set of mesh interfaces. We introduce the following discrete space:

$$(4.1) \quad \mathbf{X}_h := \left\{ \mathbf{F} \in \prod_{i=1}^N \mathcal{C}^0(\bar{\Omega}_i), \mid \forall K \in \mathcal{T}_h, \mathbf{F}|_K \in \mathbb{P}^{\ell-1} \right\}$$

where $\mathbb{P}^{\ell-1}$ denotes the vector space of vector-valued polynomial of total degree at most $\ell-1$, $\ell \geq 2$. Note that the approximation space is non-conforming, i.e., $\mathbf{X}_h \not\subset \mathbf{H}_{0,\text{curl}}(\Omega)$ and $\mathbf{X}_h \not\subset \mathbf{H}_{\text{div}}(\Omega, \epsilon)$. We assume that the mesh family is such that there exists a family of local approximation operators $\mathcal{C}_h^l : \prod_{i=1}^N \mathbf{H}^\ell(\Omega_i) \rightarrow \mathbf{X}_h$ satisfying the following properties: there exists c uniform in h such that

$$(4.2) \quad \|\mathcal{C}_h^l \mathbf{F}\|_{\mathbf{H}^r(\Omega_i)} \leq c \|\mathbf{F}\|_{\mathbf{H}^r(\Omega_i)}, \quad 0 \leq r < \frac{3}{2},$$

$$(4.3) \quad \|\mathcal{C}_h^l \mathbf{F} - \mathbf{F}\|_{\mathbf{H}^t(\Omega_i)} \leq c h^{r-t} \|\mathbf{F}\|_{\mathbf{H}^r(\Omega_i)}, \quad 0 \leq t \leq r \leq \ell, \quad t < \frac{3}{2},$$

for every $\mathbf{F} \in \prod_{i=1}^N \mathbf{H}^\ell(\Omega_i)$. We furthermore assume that the non-conformity across the interface Σ is not too severe in the sense that there exists a family of discrete subspaces $\mathbf{Y}_h \subset \mathbf{X}_h \cap \mathbf{H}_0^1(\Omega)$ and a family of global approximation operators $\mathcal{C}_h^g : \mathcal{C}_0^\infty(\Omega) \rightarrow \mathbf{Y}_h$ so that

$$(4.4) \quad \|\mathcal{C}_h^g \mathbf{F} - \mathbf{F}\|_{\mathbf{H}^t(\Omega)} \leq c h^{r-t} \|\mathbf{F}\|_{\mathbf{H}^r(\Omega)}, \quad 0 \leq t \leq r \leq \ell, \quad t < \frac{3}{2},$$

for every $\mathbf{F} \in \mathcal{C}_0^\infty(\Omega)$. An example of triangulation satisfying the above geometric assumption is shown in Figure 1. We additionally introduce the scalar-valued discrete space

$$(4.5) \quad M_h := \{q \in C^0(\bar{\Omega}), \mid \forall K \in \mathcal{T}_h, q \in \mathbb{P}^{\ell-1}, q|_\Gamma = 0\} \subset H_0^1(\Omega).$$

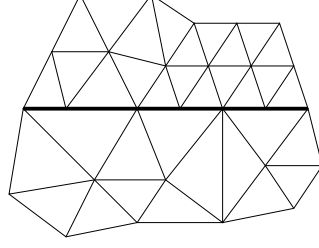


FIGURE 1. Example of an admissible mesh. The interface Σ is materialized by the thick line.

Again, the approximation theory of finite elements ensures that there exists an approximation operator $\mathcal{C}_h^p : H_0^1(\Omega) \rightarrow M_h$ satisfying the scalar counterparts of (4.2) and (4.3) for all $q \in H_0^1(\Omega) \cap H^l(\Omega)$.

$$(4.6) \quad \|\mathcal{C}_h^p q\|_{H^l(\Omega)} \leq c \|q\|_{H^l(\Omega)} \quad 0 \leq l \leq \frac{3}{2}$$

$$(4.7) \quad \|\mathcal{C}_h^p q - q\|_{H^t(\Omega)} \leq ch^{l-t} \|q\|_{\mathbf{H}^l(\Omega)} \quad 0 \leq t \leq l \leq \ell, \quad t < \frac{3}{2}.$$

Note that the possibility of constructing M_h and \mathcal{C}_h^p is compatible with the existence of \mathbf{Y}_h .

We denote \mathcal{F}_h^i the set of the mesh interfaces: F is an interface if there are two elements in \mathcal{T}_h , say K_m and K_n so that $F = K_m \cap K_n$ and F is a $d-1$ manifold. We denote \mathcal{F}_h^∂ the set of the boundary faces: F is a boundary face if there is an element in \mathcal{T}_h , say K_i so that $F = K_i \cap \Gamma$ and F is a $d-1$ manifold. To simplify notation we also introduce $\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^\partial$. For any mesh interface $F \in \mathcal{F}_h^i$, $F = K_m \cap K_n$ and any function \mathbf{v} whose restrictions over K_m and K_n are continuous, we define the tangent and normal jump of \mathbf{v} across F by

$$(4.8) \quad \llbracket \mathbf{v} \times \mathbf{n} \rrbracket(\mathbf{x}) := (\mathbf{v}|_{K_m} \times \mathbf{n}_m)(\mathbf{x}) + (\mathbf{v}|_{K_n} \times \mathbf{n}_n)(\mathbf{x}), \quad \forall \mathbf{x} \in F,$$

$$(4.9) \quad \llbracket \mathbf{v} \cdot \mathbf{n} \rrbracket(\mathbf{x}) := (\mathbf{v}|_{K_m} \cdot \mathbf{n}_m)(\mathbf{x}) + (\mathbf{v}|_{K_n} \cdot \mathbf{n}_n)(\mathbf{x}), \quad \forall \mathbf{x} \in F,$$

where \mathbf{n}_l is the unit outer normal to K_l . The average of \mathbf{v} across across F is defined by

$$(4.10) \quad \{\!\!\{ \mathbf{v} \}\!\!\}(\mathbf{x}) := \frac{1}{2} (\mathbf{v}|_{K_m}(\mathbf{x}) + \mathbf{v}|_{K_n}(\mathbf{x})), \quad \forall \mathbf{x} \in F.$$

Whenever F is a boundary face we set $\llbracket \mathbf{v} \times \mathbf{n} \rrbracket(\mathbf{x}) := \mathbf{v}|_{K_m} \times \mathbf{n}_m(\mathbf{x})$, $\llbracket \mathbf{v} \cdot \mathbf{n} \rrbracket(\mathbf{x}) := \mathbf{v}|_{K_m} \cdot \mathbf{n}_m(\mathbf{x})$ and $\{\!\!\{ \mathbf{v} \}\!\!\}(\mathbf{x}) := \mathbf{v}|_{K_m}(\mathbf{x})$.

Remark 4.1. Note that for any $\mathbf{F} \in \mathcal{C}_0^\infty(\Omega)$, $\mathcal{C}_h^g \mathbf{F} \in \mathbf{H}_0^1(\Omega)$; in particular, we have $\llbracket \mathcal{C}_h^g \mathbf{F} \times \mathbf{n} \rrbracket = 0$ across all the interfaces in \mathcal{F}_h^i .

4.2. Discrete formulation. It will be useful to work with broken norms; for instance, we introduce the following notation:

$$(4.11) \quad \|v\|_{H^s(\Omega_\Sigma)}^2 := \sum_{i=1}^N \|v\|_{H^s(\Omega_i)}^2, \quad (v, w)_{\Omega_\Sigma} := \sum_{i=1}^N \int_{\Omega_i} vw,$$

$$(4.12) \quad \|v\|_{L^2(\Sigma \cup \Gamma)}^2 := \|v\|_{L^2(\Sigma)}^2 + \|v\|_{L^2(\Gamma)}^2, \quad (v, w)_{\Sigma \cup \Gamma} := \int_{\Sigma} vw + \int_{\Gamma} vw.$$

We construct a discrete formulation of (2.10) by proceeding as in Bonito and Guermond [4]. Let $\alpha \in [0, 1]$ be a parameter yet to be chosen. We define the following bilinear form $a_h : \mathbf{X}_h \times \mathbf{M}_h \longrightarrow \mathbb{R}$,

$$\begin{aligned}
 a_h((\mathbf{E}_h, p_h), (\mathbf{F}_h, q_h)) &:= (\kappa \nabla \times \mathbf{E}_h, \nabla \times \mathbf{F}_h)_{\Omega_\Sigma} + (\{\kappa \nabla \times \mathbf{E}_h\}, [\mathbf{F}_h \times \mathbf{n}])_{\Sigma \cup \Gamma} \\
 &\quad + \theta (\{\kappa \nabla \times \mathbf{F}_h\}, [\mathbf{E}_h \times \mathbf{n}])_{\Sigma \cup \Gamma} + \gamma h^{-1} (\{\kappa\} [\mathbf{E}_h \times \mathbf{n}], [\mathbf{F}_h \times \mathbf{n}])_{\Sigma \cup \Gamma} \\
 (4.13) \quad &\quad + (\varepsilon \nabla p_h, \mathbf{F}_h)_\Omega - (\varepsilon \mathbf{E}_h, \nabla q_h)_\Omega + c_\alpha \left(h^{2\alpha} (\nabla \cdot (\varepsilon \mathbf{E}_h), \nabla \cdot (\varepsilon \mathbf{F}_h))_{\Omega_\Sigma} \right. \\
 &\quad \left. + h^{2(1-\alpha)} (\varepsilon \nabla p_h, \nabla q_h)_\Omega + h^{(2\alpha-1)} ([\varepsilon \mathbf{E}_h \cdot \mathbf{n}], [\varepsilon \mathbf{F}_h \cdot \mathbf{n}])_\Sigma \right),
 \end{aligned}$$

where $\gamma, c_\alpha > 0$, and $\theta \in \{-1, 0, +1\}$ are user-defined parameters. We say that the formulation is anti-symmetric, incomplete, or symmetric depending whether θ is equal to $-1, 0$, or 1 , respectively. The choice $\theta = 1$ ensures the adjoint consistency of the method. The term proportional to γ enforces the weak continuity of the tangent component of \mathbf{E} . The purpose of the term proportional to c_α is to penalize $\nabla \cdot (\varepsilon \mathbf{E}_h)$ in $\mathbf{H}^{-\alpha}(\Omega)$. The exponent α is somewhat similar to the exponent that is used in Costabel and Dauge [12] to define the L^2 -weighted space that controls $\nabla \cdot (\varepsilon \mathbf{E}_h)$.

The discrete formulation considered in the rest of the paper consists of looking for $(\mathbf{E}_h, p_h) \in \mathbf{X}_h \times \mathbf{M}_h$ such that the following holds for all $(\mathbf{F}_h, q_h) \in \mathbf{X}_h \times \mathbf{M}_h$:

$$(4.14) \quad a_h((\mathbf{E}_h, p_h), (\mathbf{F}_h, q_h)) = (\varepsilon \mathbf{g}, \mathbf{F}_h)_\Omega + c_\alpha h^{2(1-\alpha)} (\varepsilon \mathbf{g}, \nabla q_h)_\Omega,$$

where $(\cdot, \cdot)_D$ henceforth denotes the scalar product in $L^2(D)$.

To perform the consistency analysis of the method we are led to introduce

$$(4.15) \quad \mathbf{Z}^s(\Omega) = \{\mathbf{F} \in \mathbf{H}_{0,\text{curl}}^s(\Omega); \nabla \times (\kappa \nabla \times \mathbf{F}) \in \mathbf{L}^2(\Omega), \nabla \cdot (\varepsilon \mathbf{F}) \in \mathbf{L}^2(\Omega)\}.$$

Owing to Theorem 2.1, it is a priori known that there exists $s > 0$ such that the solution to the boundary value problem (2.10) is in $\mathbf{Z}^s(\Omega)$. We shall use the notation \mathbf{Z}^s instead of $\mathbf{Z}^s(\Omega)$ when no confusion is possible.

Proposition 4.1. *Assuming (2.13), it is possible to extend the bilinear form $a_h(\cdot, \cdot)$ to $[(\mathbf{Z}^s + \mathbf{X}_h) \times H_0^1(\Omega)]^2$ for all $s > 0$.*

Proof. Note first that $M_h \subset H_0^1(\Omega)$ and the extension of the bilinear form to scalar fields in $H_0^1(\Omega)$ does not pose any difficulty. We decompose a_h into three pieces:

$$\begin{aligned}
 a_{0h}((\mathbf{E}_h, p_h), (\mathbf{F}_h, q_h)) &:= (\{\kappa \nabla \times \mathbf{E}_h\}, [\mathbf{F}_h \times \mathbf{n}])_{\Sigma \cup \Gamma} + \theta (\{\kappa \nabla \times \mathbf{F}_h\}, [\mathbf{E}_h \times \mathbf{n}])_{\Sigma \cup \Gamma} \\
 a_{1h}((\mathbf{E}_h, p_h), (\mathbf{F}_h, q_h)) &:= (\kappa \nabla \times \mathbf{E}_h, \nabla \times \mathbf{F}_h)_{\Omega_\Sigma} + \gamma h^{-1} (\{\kappa\} [\mathbf{E}_h \times \mathbf{n}], [\mathbf{F}_h \times \mathbf{n}])_{\Sigma \cup \Gamma} \\
 a_{2h}((\mathbf{E}_h, p_h), (\mathbf{F}_h, q_h)) &:= c_\alpha \left(h^{2\alpha} (\nabla \cdot (\varepsilon \mathbf{E}_h), \nabla \cdot (\varepsilon \mathbf{F}_h))_{\Omega_\Sigma} + h^{2(1-\alpha)} (\varepsilon \nabla p_h, \nabla q_h)_\Omega \right. \\
 &\quad \left. + h^{(2\alpha-1)} ([\varepsilon \mathbf{E}_h \cdot \mathbf{n}], [\varepsilon \mathbf{F}_h \cdot \mathbf{n}])_\Sigma \right) + (\varepsilon \nabla p_h, \mathbf{F}_h)_\Omega - (\varepsilon \mathbf{E}_h, \nabla q_h)_\Omega.
 \end{aligned}$$

The bilinear form a_{1h} can clearly be extended to $[(\mathbf{Z}^s + \mathbf{X}_h) \times H_0^1(\Omega)]^2$, since every function \mathbf{E} in \mathbf{Z}^s is such that $[\mathbf{E} \times \mathbf{n}]_{\Sigma \cup \Gamma}$ is zero. Hence, if either $(\mathbf{E}, \mathbf{F}) \in \mathbf{Z}^s \times (\mathbf{Z}^s + \mathbf{X}_h)$ or $(\mathbf{E}, \mathbf{F}) \in (\mathbf{Z}^s + \mathbf{X}_h) \times \mathbf{Z}^s$, we set

$$a_{1h}((\mathbf{E}, p), (\mathbf{F}, q)) := (\kappa \nabla \times \mathbf{E}, \nabla \times \mathbf{F})_{\Omega_\Sigma},$$

for all $(p, q) \in H_0^1(\Omega)$. The bilinear form a_{2h} can also be extended to $[(\mathbf{Z}^s + \mathbf{X}_h) \times H_0^1(\Omega)]^2$, since every function \mathbf{E} in \mathbf{Z}^s is such that $[\varepsilon \mathbf{E} \cdot \mathbf{n}]_{\Sigma \cup \Gamma}$ is zero. Hence, if either

$(\mathbf{E}, \mathbf{F}) \in \mathbf{Z}^s \times (\mathbf{Z}^s + \mathbf{X}_h)$ or $(\mathbf{E}, \mathbf{F}) \in (\mathbf{Z}^s + \mathbf{X}_h) \times \mathbf{Z}^s$, we set

$$a_{2h}((\mathbf{E}, p), (\mathbf{F}, q)) := c_\alpha \left(h^{2\alpha} (\nabla \cdot (\varepsilon \mathbf{E}), \nabla \cdot (\varepsilon \mathbf{F}))_{\Omega_\Sigma} + h^{2(1-\alpha)} (\varepsilon \nabla p, \nabla q)_\Omega \right) \\ + (\varepsilon \nabla p, \mathbf{F})_\Omega - (\varepsilon \mathbf{E}, \nabla q)_\Omega.$$

for all $(p, q) \in H_0^1(\Omega)$.

The question of the extension of a_{0h} is more subtle, and we must now distinguish the trial and test spaces. We are going to use Lemma A.3 that shows that the bilinear form $(\mathbf{H}^s(\Omega) \cap \mathbf{H}_{\text{curl}}(\Omega)) \times \mathbf{X}_h \ni (\phi, \mathbf{F}_h) \mapsto \int_F \phi \cdot (\mathbf{F}_h \times \mathbf{n}) \in \mathbb{R}$ is well defined for all $F \in \mathcal{F}_h$. Let \mathbf{E} be a member of \mathbf{Z}^s , then $\nabla \times \mathbf{E} \in \mathbf{H}^s(\Omega)$, $s > 0$ and in particular, $\nabla \times \mathbf{E} \in \mathbf{H}^\sigma(\Omega)$ for some $\sigma \in (0, \frac{1}{2})$. Owing to (2.13), κ can be shown to be in $W_\Sigma^{1,\infty}(\Omega)$ and $\kappa \nabla \times \mathbf{E} \in \mathbf{H}^\sigma(\Omega)$, see e.g. Bonito et al. [5]. Note in addition that \mathbf{E} being a member of \mathbf{Z}^s implies that $\nabla \times (\kappa \nabla \times \mathbf{E}) \in \mathbf{L}^2(\Omega)$, which in turn also implies that $\llbracket \kappa \nabla \times \mathbf{E} \rrbracket_\Sigma = \kappa \nabla \times \mathbf{E}|_\Sigma$. Hence, Lemma A.3 is used to justify the expression $\int_F \kappa \nabla \times \mathbf{E} \cdot (\mathbf{F}_h \times \mathbf{n})$ for all $F \in \mathcal{F}_h$ and for all $(\mathbf{E}, \mathbf{F}_h) \in \mathbf{Z}^s \times \mathbf{X}_h$. The extension of a_{0h} for $(\mathbf{E}_h, \mathbf{F}) \in \mathbf{X}_h \times \mathbf{Z}^s$ is justified similarly. The extension of a_{0h} for $(\mathbf{E}, \mathbf{F}) \in \mathbf{Z}^s \times \mathbf{Z}^s$ is trivial since the tangent jumps of \mathbf{E} and \mathbf{F} across F are zero. Summing up, a_{0h} can be extended to $[(\mathbf{Z}^s + \mathbf{X}_h) \times H_0^1(\Omega)]^2$ by setting

$$a_{0h}((\mathbf{E} + \mathbf{E}_h, p), (\mathbf{F} + \mathbf{F}_h, q)) := (\kappa \nabla \times \mathbf{E}, \llbracket \mathbf{F}_h \times \mathbf{n} \rrbracket)_{\Sigma \cup \Gamma} + (\llbracket \kappa \nabla \times \mathbf{E}_h \rrbracket, \llbracket \mathbf{F}_h \times \mathbf{n} \rrbracket)_{\Sigma \cup \Gamma} \\ + \theta (\kappa \nabla \times \mathbf{F}, \llbracket \mathbf{E}_h \times \mathbf{n} \rrbracket)_{\Sigma \cup \Gamma} + \theta (\llbracket \kappa \nabla \times \mathbf{F}_h \rrbracket, \llbracket \mathbf{E}_h \times \mathbf{n} \rrbracket)_{\Sigma \cup \Gamma},$$

for all $(\mathbf{E}, \mathbf{E}_h) \in \mathbf{Z}^s \times \mathbf{X}_h$, all $(\mathbf{F}, \mathbf{F}_h) \in \mathbf{Z}^s \times \mathbf{X}_h$, and all $(p, q) \in H_0^1(\Omega)$. This ends the proof. \square

Remark 4.1. One could avoid invoking Lemma A.3 in the above proof by using instead a result from Buffa and Ciarlet [8] where it is shown that the bilinear form $\mathbf{H}_{\text{curl}}(\Omega) \times \mathbf{H}_{\text{curl}}(\Omega) \ni (\phi, \mathbf{F}) \mapsto \int_F \phi \cdot (\mathbf{F} \times \mathbf{n}) \in \mathbb{R}$ is well defined and continuous for all $F \in \mathcal{F}_h$.

Lemma 4.1. Assume (2.13) and let (\mathbf{E}, p) be the solution of (2.10). Let $s > 0$ be such that $\mathbf{E} \in \mathbf{Z}^s$. The following holds for any $(\mathbf{F} + \mathbf{F}_h, q) \in (\mathbf{Z}^s + \mathbf{X}_h) \times H_0^1(\Omega)$:

$$a_h((\mathbf{E}, p), (\mathbf{F} + \mathbf{F}_h, q)) = (\varepsilon \mathbf{g}, \mathbf{F} + \mathbf{F}_h)_\Omega + c_\alpha h^{2(1-\alpha)} (\varepsilon \mathbf{g}, \nabla q)_\Omega.$$

Proof. Let us first observe that

$$a_h((\mathbf{E}, p), (\mathbf{F} + \mathbf{F}_h, q)) = (\kappa \nabla \times \mathbf{E}, \nabla \times (\mathbf{F} + \mathbf{F}_h))_{\Omega_\Sigma} + (\kappa \nabla \times \mathbf{E}, \llbracket \mathbf{F}_h \times \mathbf{n} \rrbracket)_{\Sigma \cup \Gamma} \\ + (\varepsilon \nabla p, \mathbf{F} + \mathbf{F}_h)_\Omega + c_\alpha h^{2(1-\alpha)} (\varepsilon \nabla p, \nabla q)_\Omega,$$

where all the terms make sense owing to the extension properties of a_h stated in Proposition 4.1. We now test (2.10) with $\mathbf{F} + \mathbf{F}_h \in (\mathbf{Z}^s + \mathbf{X}_h)$,

$$(\nabla \times (\kappa \nabla \times \mathbf{E}), \mathbf{F})_\Omega + \sum_{i=1}^N (\nabla \times (\kappa \nabla \times \mathbf{E}), \mathbf{F}_h)_{\Omega_i} + (\varepsilon \nabla p, \mathbf{F} + \mathbf{F}_h)_\Omega = (\varepsilon \mathbf{g}, \mathbf{F} + \mathbf{F}_h)_\Omega,$$

and we perform the integration by parts over Ω when the test function is \mathbf{F} and over each sub-domain when the test function is \mathbf{F}_h ,

$$(\kappa \nabla \times \mathbf{E}, \nabla \times \mathbf{F})_\Omega + \sum_{i=1}^N (\kappa \nabla \times \mathbf{E}, \nabla \times \mathbf{F}_h)_{\Omega_i} + (\kappa \nabla \times \mathbf{E}, \llbracket \mathbf{F}_h \times \mathbf{n} \rrbracket)_{\Sigma \cup \Gamma} \\ + (\varepsilon \nabla p, \mathbf{F} + \mathbf{F}_h)_\Omega = (\varepsilon \mathbf{g}, \mathbf{F} + \mathbf{F}_h)_\Omega.$$

Note that the term $(\kappa \nabla \times \mathbf{E}, [\mathbf{F}_h \times \mathbf{n}])_{\Sigma \cup \Gamma}$ is meaningful owing to Lemma A.3 and \mathbf{E} being a member of \mathbf{Z}^s . This implies that

$$a_h((\mathbf{E}, p), (\mathbf{F} + \mathbf{F}_h, q)) = (\varepsilon \mathbf{g}, \mathbf{F} + \mathbf{F}_h)_\Omega + c_\alpha h^{2(1-\alpha)} (\varepsilon \nabla p, \nabla q)_\Omega.$$

Upon testing again (2.10) with ∇q , $q \in H_0^1(\Omega)$, we infer that $(\varepsilon \nabla p, \nabla q)_\Omega = (\varepsilon \mathbf{g}, \nabla q)_\Omega$, which in turn implies the desired result. \square

Remark 4.2 (Continuous Approximation of p). Observe that the approximation of the Lagrange multiplier p is globally continuous. This is critical to derive a global control of $\nabla \cdot (\varepsilon \mathbf{E})$ in $\mathbf{H}^{-\alpha}(\Omega)$ instead of $\prod_{i=1}^N \mathbf{H}^{-\alpha}(\Omega_i)$.

4.3. Well posedness of the discrete formulation. We discuss in this section the existence and uniqueness of a solution (\mathbf{E}_h, p_h) to (4.14). This issue is addressed by equipping $\mathbf{X}_h \times \mathbf{M}_h$ with the following discrete norm:

$$(4.16) \quad \begin{aligned} \|\mathbf{F}_h, q_h\|_h^2 := & \|\kappa^{\frac{1}{2}} \nabla \times \mathbf{F}_h\|_{L^2(\Omega_\Sigma)}^2 + \gamma h^{-1} \|\llbracket \kappa \rrbracket^{\frac{1}{2}} [\mathbf{F}_h \times \mathbf{n}]\|_{L^2(\Sigma \cup \Gamma)}^2 \\ & + c_\alpha \left(h^{2\alpha} \|\nabla \cdot (\varepsilon \mathbf{F}_h)\|_{L^2(\Omega_\Sigma)}^2 + h^{2(1-\alpha)} \|\varepsilon^{\frac{1}{2}} \nabla q_h\|_{L^2(\Omega)}^2 \right. \\ & \left. + h^{(2\alpha-1)} \|\llbracket \varepsilon \mathbf{F}_h \cdot \mathbf{n} \rrbracket\|_{L^2(\Sigma)}^2 \right), \end{aligned}$$

by proving a coercivity property, uniform in h , and by establishing some continuity estimates for the bilinear form $a_h(\cdot, \cdot)$. We first establish the coercivity of a_h .

Proposition 4.2 (Coercivity). *If $\theta \in \{0, 1\}$, there exists $\gamma_0 > 0$ and $c(\gamma_0) > 0$, uniform with respect to h , so that the following holds for all $\gamma \geq \gamma_0$ and for any $0 \leq \alpha \leq 1$:*

$$(4.17) \quad a_h((\mathbf{E}_h, p_h), (\mathbf{E}_h, p_h)) \geq c(\gamma_0) \|\mathbf{E}_h, p_h\|_h^2, \quad \forall (\mathbf{E}_h, p_h) \in \mathbf{X}_h \times M_h,$$

and this inequality holds for all $\gamma > 0$ with $c(\gamma_0) = 1$ if $\theta = -1$.

Proof. We first observe that

$$a_h((\mathbf{E}_h, p_h), (\mathbf{E}_h, p_h)) = \|\mathbf{E}_h, p_h\|_h^2 + (1 + \theta) (\llbracket \kappa \nabla \times \mathbf{E}_h \rrbracket, [\mathbf{E}_h \times \mathbf{n}])_{\Sigma \cup \Gamma}.$$

The conclusion is evident if $\theta = -1$. Otherwise we have to control the term $(\llbracket \kappa \nabla \times \mathbf{E}_h \rrbracket, [\mathbf{E}_h \times \mathbf{n}])_{\Sigma \cup \Gamma}$. Invoking a trace and a Young inequality we obtain

$$(\llbracket \kappa \nabla \times \mathbf{F}_h \rrbracket, [\mathbf{F}_h \times \mathbf{n}])_{\Sigma \cup \Gamma} \leq \frac{1}{4} \|\kappa^{\frac{1}{2}} \nabla \times \mathbf{F}_h\|_{L^2(\Omega_\Sigma)}^2 + c_0 h^{-1} \|\llbracket \kappa \rrbracket^{\frac{1}{2}} [\mathbf{F}_h \times \mathbf{n}]\|_{L^2(\Sigma \cup \Gamma)}^2.$$

Hence, if $\gamma \geq \gamma_0 := 4c_0$, we infer that the following holds:

$$(4.18) \quad a_h((\mathbf{E}_h, p_h), (\mathbf{E}_h, p_h)) \geq \frac{1}{2} \|\mathbf{E}_h, p_h\|_h^2 \geq 0.$$

This completes the proof. \square

We now establish the uniform boundedness of the bilinear form a_h .

Proposition 4.3 (Continuity). *For any $s \in (0, \frac{1}{2})$, there is $c > 0$, uniform in h such that the following holds for any $0 \leq \alpha \leq 1$ and for every $(\mathbf{E}, p) \in \mathbf{Z}^s \times H_0^1(\Omega)$ and $(\mathbf{G}_h, d_h), (\mathbf{F}_h, q_h) \in \mathbf{X}_h \times M_h$:*

$$(4.19) \quad \begin{aligned} c \frac{a_h((\mathbf{E} - \mathbf{G}_h, p - d_h), (\mathbf{F}_h, q_h))}{\|\mathbf{F}_h, q_h\|_h} \leq & \|\mathbf{E} - \mathbf{G}_h\|_h + h^{\alpha-1} \|\mathbf{E} - \mathbf{G}_h\|_{L^2(\Omega)} \\ & + h^\sigma \|\kappa \nabla \times (\mathbf{E} - \mathbf{G}_h)\|_{\mathbf{H}^\sigma(\mathcal{T}_h)} + h \|\nabla \times \kappa \nabla \times (\mathbf{E} - \mathbf{G}_h)\|_{L^2(\mathcal{T}_h)} \\ & + h^{-\alpha} \|p - d_h\|_{L^2(\Omega)} + h^{(\frac{1}{2}-\alpha)} \|p - d_h\|_{L^2(\Sigma)}. \end{aligned}$$

where $\|\cdot\|_{\mathbf{L}^2(\mathcal{T}_h)}^2 := \sum_{K \in \mathcal{T}_h} \|\cdot\|_{\mathbf{L}^2(K)}^2$.

Proof. Upon applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} & (\kappa \nabla \times (\mathbf{E} - \mathbf{G}_h), \nabla \times \mathbf{F}_h)_{\Omega_\Sigma} + \gamma h^{-1} (\llbracket \kappa \rrbracket \llbracket (\mathbf{E} - \mathbf{G}_h) \times \mathbf{n} \rrbracket, \llbracket \mathbf{F}_h \times \mathbf{n} \rrbracket)_{\Sigma \cup \Gamma} \\ & + c_\alpha \left(h^{2\alpha} (\nabla \cdot (\varepsilon (\mathbf{E} - \mathbf{G}_h)), \nabla \cdot (\varepsilon \mathbf{F}_h))_{\Omega_\Sigma} + h^{2(1-\alpha)} (\varepsilon \nabla (p - d_h), \nabla q_h)_\Omega \right. \\ & \left. + h^{(2\alpha-1)} (\llbracket \varepsilon (\mathbf{E} - \mathbf{G}_h) \cdot \mathbf{n} \rrbracket, \llbracket \varepsilon \mathbf{F}_h \cdot \mathbf{n} \rrbracket)_\Sigma \right), \\ & \leq \|\mathbf{F}_h, q_h\|_h \|\mathbf{E} - \mathbf{G}_h, p - d_h\|_h. \end{aligned}$$

We now bound separately the remaining terms appearing in the definition (4.13) of $a_h(\cdot, \cdot)$:

$$\begin{aligned} -(\varepsilon (\mathbf{E} - \mathbf{G}_h), \nabla q_h)_\Omega & \leq \varepsilon_{\max} h^{\alpha-1} \|\nabla q_h\|_{\mathbf{L}^2(\Omega)} h^{1-\alpha} \|\mathbf{E} - \mathbf{G}_h\|_{\mathbf{L}^2(\Omega)}, \\ (\varepsilon \nabla (p - d_h), \mathbf{F}_h)_\Omega & \leq h^\alpha \|\nabla \cdot (\varepsilon \mathbf{F}_h)\|_{L^2(\Omega_\Sigma)} h^{-\alpha} \|p - d_h\|_{L^2(\Omega)} \\ & \quad + h^{(\alpha-\frac{1}{2})} \|\llbracket \varepsilon \mathbf{F}_h \cdot \mathbf{n} \rrbracket\|_{\mathbf{L}^2(\Sigma)} h^{(\frac{1}{2}-\alpha)} \|p - d_h\|_{L^2(\Sigma)}, \end{aligned}$$

where we used an integration by parts for the second estimate. We are now left with the consistency terms

$$(4.20) \quad (\llbracket \kappa \nabla \times (\mathbf{E} - \mathbf{G}_h) \rrbracket, \llbracket \mathbf{F}_h \times \mathbf{n} \rrbracket)_{\Sigma \cup \Gamma} + \theta (\llbracket \kappa \nabla \times \mathbf{F}_h \rrbracket, \llbracket (\mathbf{E} - \mathbf{G}_h) \times \mathbf{n} \rrbracket)_{\Sigma \cup \Gamma}.$$

For the first term in (4.20), we apply Lemma A.3 with $\mathbf{v} = \llbracket \mathbf{F}_h \times \mathbf{n} \rrbracket$, which is a polynomial of degree $\ell - 1$, and $\phi = \llbracket \kappa \nabla \times (\mathbf{E} - \mathbf{G}_h) \rrbracket$. Then for any $F \in \mathcal{F}_h$, we infer that

$$\begin{aligned} & (\llbracket \kappa \nabla \times (\mathbf{E} - \mathbf{G}_h) \rrbracket, \llbracket \mathbf{F}_h \times \mathbf{n} \rrbracket)_F \leq ch^{-\frac{1}{2}} \|\llbracket \mathbf{F}_h \times \mathbf{n} \rrbracket\|_{\mathbf{L}^2(F)} \\ & \quad \times \sum_{i=1}^2 \left(h^s \|\kappa \nabla \times (\mathbf{E} - \mathbf{G}_h)\|_{\mathbf{H}^s(K_i)} + h \|\nabla \times \kappa \nabla \times (\mathbf{E} - \mathbf{G}_h)\|_{\mathbf{L}^2(K_i)} \right. \\ & \quad \left. + \|\kappa \nabla \times (\mathbf{E} - \mathbf{G}_h)\|_{\mathbf{L}^2(K_i)} \right), \end{aligned}$$

where $K_1, K_2 \in \mathcal{T}_h$ such that $F = \overline{K_1} \cap \overline{K_2}$. Hence, summing over all the faces we arrive at

$$\begin{aligned} & (\llbracket \kappa \nabla \times (\mathbf{E} - \mathbf{G}_h) \rrbracket, \llbracket \mathbf{F}_h \times \mathbf{n} \rrbracket)_{\Sigma \cup \Gamma} \leq ch^{-\frac{1}{2}} \|\llbracket \mathbf{F}_h \times \mathbf{n} \rrbracket\|_{\mathbf{L}^2(\Sigma \cup \Gamma)} \left(h^s \|\kappa \nabla \times (\mathbf{E} - \mathbf{G}_h)\|_{\mathbf{H}^s(\mathcal{T}_h)} \right. \\ & \quad \left. + h \|\nabla \times \kappa \nabla \times (\mathbf{E} - \mathbf{G}_h)\|_{\mathbf{L}^2(\mathcal{T}_h)} + \|\kappa \nabla \times (\mathbf{E} - \mathbf{G}_h)\|_{\mathbf{L}^2(\mathcal{T}_h)} \right). \end{aligned}$$

For the second term in (4.20) we notice that $\llbracket (\mathbf{E} - \mathbf{G}_h) \times \mathbf{n} \rrbracket = -\llbracket \mathbf{G}_h \times \mathbf{n} \rrbracket$ owing to the regularity of \mathbf{E} . Then by using Lemma A.3 again, we arrive at

$$\begin{aligned} & (\llbracket \kappa \nabla \times \mathbf{F}_h \rrbracket, \llbracket (\mathbf{E} - \mathbf{G}_h) \times \mathbf{n} \rrbracket)_{\Sigma \cup \Gamma} \leq ch^{-\frac{1}{2}} \|\llbracket \mathbf{G}_h \times \mathbf{n} \rrbracket\|_{L^2(\Sigma \cup \Gamma)} \|\kappa \nabla \times \mathbf{F}_h\|_{L^2(\mathcal{T}_h)} \\ & \leq ch^{-\frac{1}{2}} \|\llbracket (\mathbf{E} - \mathbf{G}_h) \times \mathbf{n} \rrbracket\|_{L^2(\Sigma \cup \Gamma)} \|\kappa \nabla \times \mathbf{F}_h\|_{L^2(\mathcal{T}_h)}, \end{aligned}$$

where we used the inverse inequalities

$$\begin{aligned} h \|\nabla \times \kappa \nabla \times \mathbf{F}_h\|_{L^2(\mathcal{T}_h)} & \leq c \|\kappa \nabla \times \mathbf{F}_h\|_{L^2(\mathcal{T}_h)}, \\ h^s \|\kappa \nabla \times \mathbf{F}_h\|_{H^s(\mathcal{T}_h)} & \leq c \|\kappa \nabla \times \mathbf{F}_h\|_{L^2(\mathcal{T}_h)}. \end{aligned}$$

The desired result is obtained by gathering the above estimates. \square

The following result will be instrumental to apply the Nitsche-Aubin duality argument and derive a convergence result in $\mathbf{L}^2(\Omega)$.

Proposition 4.4 (Adjoint continuity). *For any $s \in (0, \frac{1}{2})$, there is $c > 0$, uniform in h such that for any $0 \leq \alpha \leq 1$, the following holds for every $(\mathbf{E}, p), (\mathbf{F}, q) \in \mathbf{Z}^s \times H_0^1(\Omega)$, $\mathbf{F}_h \in \mathbf{Y}_h$, $q_h \in M_h$ and $(\mathbf{G}_h, d_h) \in \mathbf{X}_h \times M_h$:*

$$(4.21) \quad \begin{aligned} c \frac{a_h((\mathbf{E} - \mathbf{G}_h, p - d_h), (\mathbf{F} - \mathbf{F}_h, q - q_h))}{\|\mathbf{E} - \mathbf{G}_h, p - d_h\|_h} &\leq \|\mathbf{F} - \mathbf{F}_h, q - q_h\|_h + h^{\alpha-1} \|\mathbf{F} - \mathbf{F}_h\|_{\mathbf{L}^2(\Omega)} \\ &\quad + h^s \|\kappa \nabla \times (\mathbf{F} - \mathbf{F}_h)\|_{H^s(\mathcal{T}_h)} \\ &\quad + h \|\nabla \times \kappa \nabla \times (\mathbf{F} - \mathbf{F}_h)\|_{L^2(\mathcal{T}_h)} \\ &\quad + h^{-\alpha} \|q - q_h\|_{L^2(\Omega)} + h^{(\frac{1}{2}-\alpha)} \|q - q_h\|_{L^2(\Sigma)}. \end{aligned}$$

Proof. The proof proceeds similarly as in the proof of Proposition 4.3. The only difference here is that we have $(\llbracket \nabla \times (\mathbf{E} - \mathbf{G}_h) \rrbracket, \llbracket (\mathbf{F} - \mathbf{F}_h) \times \mathbf{n} \rrbracket)_{\Sigma \cup \Gamma} = 0$, owing to the assumption that $\mathbf{F}_h \in \mathbf{Y}_h \subset \mathbf{X}_h \cap \mathbf{H}_0^1(\Omega)$. This identity makes the analysis of the consistency term (4.20) tractable. \square

5. CONVERGENCE ANALYSIS FOR THE BOUNDARY VALUE PROBLEM

In the first part of this section, we prove two convergence results for the discrete problem (4.14) using the discrete norm $\|\cdot\|_h$, one assuming minimal regularity and the other assuming full smoothness. In the second part of the section we use a Nitsche-Aubin duality argument to establish convergence in $\mathbf{L}^2(\Omega)$. The performance of the method is numerically illustrated at the end of the section.

5.1. Convergence in the discrete norm. We assume first that the solution to the boundary value problem (2.10) has minimal regularity properties, and we start with the Galerkin orthogonality.

Lemma 5.1 (Galerkin Orthogonality). *Assume (2.13), then the Galerkin orthogonality holds, i.e., let (\mathbf{E}, p) be the solution of (2.10) and (\mathbf{E}_h, p_h) be the solution of (4.14), then for any $(\mathbf{F}_h, q_h) \in \mathbf{X}_h \times M_h$*

$$(5.1) \quad a_h((\mathbf{E} - \mathbf{E}_h, p - p_h), (\mathbf{F}_h, q_h)) = 0.$$

Proof. This is a direct consequence of Lemma 4.1 and formulation (4.14). \square

Theorem 5.1. *Let $\mathbf{g} \in \mathbf{L}^2(\Omega)$ and $\tau \in (0, \min(\tau_\varepsilon, \tau_\kappa))$ where τ_ε and τ_κ are defined in Theorem 2.1. Let (\mathbf{E}, p) and (\mathbf{E}_h, p_h) be the solution of (2.10) and (4.14), respectively. Then, for any $\alpha \in (\frac{\ell(1-\tau)}{\ell-\tau}, 1)$, there exists $c > 0$, uniform in h , such that*

$$(5.2) \quad \|\mathbf{E} - \mathbf{E}_h, p - p_h\|_h \leq ch^r \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)},$$

where $r = \alpha - 1 + \tau(1 - \frac{\alpha}{\ell})$ if $\nabla \cdot \mathbf{g} = 0$ and $r = \min(1 - \alpha, \alpha - 1 + \tau(1 - \frac{\alpha}{\ell}))$ otherwise.

Proof. We first recall that, owing to Theorem 2.1, we have $\mathbf{E} \in \mathbf{H}^\tau(\Omega) \cap \mathbf{H}_{0,\text{curl}}^\tau(\Omega)$, together with the estimates

$$\|\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)} + \|\nabla \times \mathbf{E}\|_{\mathbf{H}^\tau(\Omega)} + \|\nabla \times (\kappa \nabla \times \mathbf{E})\|_{\mathbf{L}^2(\Omega)} + \|\nabla p\|_{\mathbf{L}^2(\Omega)} \leq c \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}.$$

We establish (5.2) by using the triangular inequality

$$\begin{aligned} \|\mathbf{E} - \mathbf{E}_h, p - p_h\|_h &\leq \|\mathbf{E} - \mathcal{K}_\delta \mathbf{E}, 0\|_h + \|\mathcal{K}_\delta \mathbf{E} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E}, p - \mathcal{C}_h^p p\|_h \\ &\quad + \|\mathcal{C}_h \mathcal{K}_\delta \mathbf{E} - \mathbf{E}_h, \mathcal{C}_h^p p - p_h\|_h, \end{aligned}$$

for some $\delta > 0$ to be defined later, and by bounding from above the three terms separately.

Using the definition of $\|\cdot\|_h$ together with the approximation properties of \mathcal{K}_δ , cf. (3.16)-(3.17)-(3.18), we conclude that

$$\|\mathbf{E} - \mathcal{K}_\delta \mathbf{E}, 0\|_h \leq c \left(\delta^\tau \|\nabla \times \mathbf{E}\|_{\mathbf{H}^\tau(\Omega)} + h^\alpha \delta^{\tau-1} \|\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)} + h^{\alpha-\frac{1}{2}} \|\mathcal{K}_\delta \mathbf{E}\|_{\mathbf{L}^2(\Sigma)} \right).$$

Note that the estimate (3.17) is critical to obtain a bound that depends only on $\|\nabla \times \mathbf{E}\|_{\mathbf{H}^\tau(\Omega)}$ instead of $\|\mathbf{E}\|_{\mathbf{H}^{1+\tau}(\Omega)}$. To estimate the last term in the above inequality, we apply (A.5) with $\Theta = \frac{1-2\tau}{2(1-\tau)}$,

$$\begin{aligned} h^{\alpha-\frac{1}{2}} \|\mathcal{K}_\delta \mathbf{E}\|_{\mathbf{L}^2(\Sigma)} &\leq c h^{\alpha-\frac{1}{2}} \|\mathcal{K}_\delta \mathbf{E}\|_{\mathbf{H}^\tau(\Omega)}^{1-\Theta} \|\mathcal{K}_\delta \mathbf{E}\|_{\mathbf{H}^1(\Omega)}^\Theta \\ &\leq c h^{\alpha-\frac{1}{2}} \delta^{\Theta(\tau-1)} \|\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)} \leq c h^{\alpha-\frac{1}{2}} \delta^{\tau-\frac{1}{2}} \|\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)}. \end{aligned}$$

Finally, we arrive at

$$(5.3) \quad \|\mathbf{E} - \mathcal{K}_\delta \mathbf{E}, 0\|_h \leq c \left(\delta^\tau + h^\alpha \delta^{\tau-1} + h^{\alpha-\frac{1}{2}} \delta^{\tau-\frac{1}{2}} \right) \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}.$$

Let us now turn our attention to $\|\mathcal{K}_\delta \mathbf{E} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E}, p - \mathcal{C}_h^p p\|_h$. Owing to the definition of \mathcal{C}_h^g and the regularity of $\mathcal{K}_\delta \mathbf{E}$, we have $\mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E} \in \mathbf{H}_0^1(\Omega) \subset \mathbf{H}_{0,\text{curl}}(\Omega)$, so that we only have four terms to bound (the jumps of $\mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E}$ across the mesh interfaces and the tangent trace on Γ are zero, cf. Remark 4.1). Using the properties of \mathcal{K}_δ and \mathcal{C}_h^g together with (A.4) we deduce that

$$\begin{aligned} \|\kappa^{\frac{1}{2}} \nabla \times (\mathcal{K}_\delta \mathbf{E} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E})\|_{\mathbf{L}^2(\Omega)} &\leq c h^{\ell-1} \|\mathcal{K}_\delta \mathbf{E}\|_{\mathbf{H}^\ell(\Omega)} \leq c h^{\ell-1} \delta^{\tau-\ell} \|\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)}, \\ h^\alpha \|\nabla \cdot (\varepsilon (\mathcal{K}_\delta \mathbf{E} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E}))\|_{\mathbf{L}^2(\Omega_\Sigma)} &\leq c h^{\alpha+\ell-1} \|\mathcal{K}_\delta \mathbf{E}\|_{\mathbf{H}^\ell(\Omega_\Sigma)} \leq c h^{\alpha+\ell-1} \delta^{\tau-\ell} \|\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)}, \\ h^{1-\alpha} \|\varepsilon^{\frac{1}{2}} \nabla (p - \mathcal{C}_h^p p)\|_{\mathbf{L}^2(\Omega)} &\leq c h^{1-\alpha} \|p\|_{H_0^1(\Omega)}, \\ h^{\alpha-\frac{1}{2}} \|\varepsilon (\mathcal{K}_\delta \mathbf{E} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E}) \cdot \mathbf{n}\|_{\mathbf{L}^2(\Sigma)} &\leq c h^{\alpha-\frac{1}{2}} \|\mathcal{K}_\delta \mathbf{E} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E}\|_{\mathbf{L}^2(\Sigma)} \\ &\leq c h^{\alpha-\frac{1}{2}} \|\mathcal{K}_\delta \mathbf{E} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E}\|_{\mathbf{L}^2(\Omega)}^{1-\frac{1}{2\alpha}} \|\mathcal{K}_\delta \mathbf{E} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E}\|_{\mathbf{H}^\alpha(\Omega)}^{\frac{1}{2\alpha}} \\ &\leq c h^{\alpha-\frac{1}{2}} h^{\ell(1-\frac{1}{2\alpha})} h^{(\ell-\alpha)\frac{1}{2\alpha}} \|\mathcal{K}_\delta \mathbf{E}\|_{\mathbf{H}^\ell(\Omega)} \\ &\leq c h^{\alpha+\ell-1} \delta^{\tau-\ell} \|\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)}. \end{aligned}$$

When combining the above estimates, we obtain

$$(5.4) \quad \|\mathcal{K}_\delta \mathbf{E} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E}, p - \mathcal{C}_h^p p\|_h \leq c (h^{\ell-1} \delta^{\tau-\ell} + \chi h^{1-\alpha}) \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)},$$

where $\chi = 0$ if $\nabla \cdot \mathbf{g} = 0$ and $\chi = 1$ otherwise (note that $p = 0$ when $\nabla \cdot \mathbf{g} = 0$).

The last term, $\|\mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E} - \mathbf{E}_h, \mathcal{C}_h^p p - p_h\|_h$, is a little more subtle to handle. We start from the coercivity of a_h , (4.18), and use both the Galerkin orthogonality (5.1)

and the continuity of a_h , (4.19), with $s = 1 - \alpha$ to get the following estimate:

$$\begin{aligned}
& \| \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E} - \mathbf{E}_h, \mathcal{C}_h^p p - p_h \|_h \\
& \leq c \frac{a_h((\mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E} - \mathbf{E}_h, \mathcal{C}_h^p p - p_h), (\mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E} - \mathbf{E}_h, \mathcal{C}_h^p p - p_h))}{\| \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E} - \mathbf{E}_h, \mathcal{C}_h^p p - p_h \|_h} \\
& \leq c \frac{a_h((\mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E} - \mathbf{E}, \mathcal{C}_h^p p - p), (\mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E} - \mathbf{E}_h, \mathcal{C}_h^p p - p_h))}{\| \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E} - \mathbf{E}_h, \mathcal{C}_h^p p - p_h \|_h} \\
& \leq c (\| \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E} - \mathbf{E}, \mathcal{C}_h^p p - p \|_h + h^{\alpha-1} \| \mathbf{E} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E} \|_{\mathbf{L}^2(\Omega)} \\
& \quad + h^{1-\alpha} \| \kappa \nabla \times (\mathbf{E} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E}) \|_{\mathbf{H}^{1-\alpha}(\Omega)} + h^{-\alpha} \| p - \mathcal{C}_h^p p \|_{L^2(\Omega)} \\
& \quad + h \| \nabla \times \kappa \nabla \times (\mathbf{E} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E}) \|_{\mathbf{L}^2(\mathcal{T}_h)} + h^{\frac{1}{2}-\alpha} \| p - \mathcal{C}_h^p p \|_{L^2(\Sigma)}).
\end{aligned}$$

We now handle each term in the right hand side separately. Using the triangle inequality $\| \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E} - \mathbf{E}, \mathcal{C}_h^p p - p \|_h \leq \| \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E} - \mathcal{K}_\delta \mathbf{E}, \mathcal{C}_h^p p - p \|_h + \| \mathcal{K}_\delta \mathbf{E} - \mathbf{E}, 0 \|_h$ and the estimates (5.3)-(5.4), we obtain

$$\| \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E} - \mathbf{E}, \mathcal{C}_h^p p - p \|_h \leq c \left(\delta^\tau + h^\alpha \delta^{\tau-1} + h^{\alpha-\frac{1}{2}} \delta^{\tau-\frac{1}{2}} + h^{\ell-1} \delta^{\tau-\ell} + \chi h^{1-\alpha} \right) \| \mathbf{g} \|_{\mathbf{L}^2(\Omega)}.$$

Similarly, we obtain

$$\begin{aligned}
h^{\alpha-1} \| \mathbf{E} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E} \|_{\mathbf{L}^2(\Omega)} & \leq c (h^{\alpha-1} \delta^\tau + h^{\alpha+\ell-1} \delta^{\tau-\ell}) \| \mathbf{g} \|_{\mathbf{L}^2(\Omega)}, \\
h^{1-\alpha} \| \kappa \nabla \times (\mathbf{E} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E}) \|_{\mathbf{H}^{1-\alpha}(\Omega)} & \leq c (h^{1-\alpha} \delta^{\tau+\alpha-1} + h^{\ell-1} \delta^{\tau-\ell}) \| \mathbf{g} \|_{\mathbf{L}^2(\Omega)}.
\end{aligned}$$

Note that the previous computation is valid since $1 - \alpha \leq \tau$ owing to the assumption $\alpha \in \left(\frac{\ell(1-\tau)}{\ell-\tau}, 1 \right)$. For the last term involving \mathbf{E} we use the commuting property $\bar{\delta} \nabla \times \mathcal{K}_\delta \mathbf{E} = \mathcal{K}_\delta \nabla \times \mathbf{E}$, see (3.19), to derive

$$\begin{aligned}
h \| \nabla \times (\kappa \nabla \times (\mathbf{E} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E})) \|_{\mathbf{L}^2(\mathcal{T}_h)} & \leq h \| \nabla \times (\kappa \nabla \times \mathbf{E}) \|_{\mathbf{L}^2(\mathcal{T}_h)} + h \| \nabla \times (\kappa \nabla \times \mathcal{K}_\delta \mathbf{E}) \|_{\mathbf{L}^2(\mathcal{T}_h)} \\
& \quad + h \| \nabla \times (\kappa \nabla \times (\mathcal{K}_\delta \mathbf{E} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{E})) \|_{\mathbf{L}^2(\mathcal{T}_h)} \\
& \leq c (h \| \mathbf{g} \|_{\mathbf{L}^2(\Omega)} + h \| \nabla \times \mathcal{K}_\delta \mathbf{E} \|_{\mathbf{H}^1(\Omega)} + h^{\ell-1} \| \mathcal{K}_\delta \mathbf{E} \|_{\mathbf{H}^\ell(\Omega)}) \\
& \leq c (h \| \mathbf{g} \|_{\mathbf{L}^2(\Omega)} + h \| \mathcal{K}_\delta \nabla \times \mathbf{E} \|_{\mathbf{H}^1(\Omega)} + h^{\ell-1} \delta^{\tau-\ell} \| \mathbf{E} \|_{\mathbf{H}^\tau(\Omega)}) \\
& \leq c (h + h \delta^{\tau-1} + h^{\ell-1} \delta^{\tau-\ell}) \| \mathbf{g} \|_{\mathbf{L}^2(\Omega)}.
\end{aligned}$$

For the remaining terms involving p , we use (A.4) together with the approximation properties of \mathcal{C}_h^p :

$$\begin{aligned}
h^{-\alpha} \| p - \mathcal{C}_h^p p \|_{L^2(\Omega)} & \leq c h^{1-\alpha} \| p \|_{H_0^1(\Omega)} \leq c \chi h^{1-\alpha} \| \mathbf{g} \|_{\mathbf{L}^2(\Omega)}, \\
h^{\frac{1}{2}-\alpha} \| p - \mathcal{C}_h^p p \|_{L^2(\Sigma)} & \leq h^{\frac{1}{2}-\alpha} \| p - \mathcal{C}_h^p p \|_{L^2(\Omega)}^{1-\frac{1}{2\alpha}} \| p - \mathcal{C}_h^p p \|_{H^\alpha(\Omega)}^{\frac{1}{2\alpha}} \\
& \leq c h^{\frac{1}{2}-\alpha} h^{1-\frac{1}{2\alpha}} h^{(1-\alpha)\frac{1}{2\alpha}} \| p \|_{H_0^1(\Omega)} \leq c \chi h^{1-\alpha} \| \mathbf{g} \|_{\mathbf{L}^2(\Omega)}.
\end{aligned}$$

Gathering all the above estimates together with (5.3) and (5.4), we finally obtain

$$\begin{aligned}
(5.5) \quad \| \mathbf{E} - \mathbf{E}_h, p - p_h \|_h & \leq c (\delta^\tau + \chi h^{1-\alpha} + h + h \delta^{\tau-1} + h^{\ell-1} \delta^{\tau-\ell} + h^{\alpha-1} \delta^\tau \\
& \quad + h^{1-\alpha} \delta^{\tau+\alpha-1} + h^\alpha \delta^{\tau-1} + h^{\alpha-\frac{1}{2}} \delta^{\tau-\frac{1}{2}}) \| \mathbf{g} \|_{\mathbf{L}^2(\Omega)}.
\end{aligned}$$

We want to use $\delta = h^\beta$ for some $\beta \in (0, 1)$, i.e., $\delta h^{-1} \rightarrow +\infty$ as $h \rightarrow 0$. Once the negligible terms are removed in (5.5), we derive the following estimate:

$$\| \mathbf{E} - \mathbf{E}_h, p - p_h \|_h \leq c (h^{\alpha-1} \delta^\tau + \chi h^{1-\alpha} + h^{\ell-1} \delta^{\tau-\ell}) \| \mathbf{g} \|_{\mathbf{L}^2(\Omega)}.$$

Using $\delta = h^{1-\frac{\alpha}{\ell}}$ implies that $h^{\alpha-1}\delta^\tau = h^{\ell-1}\delta^{\tau-\ell}$ and we arrive at

$$\|\mathbf{E} - \mathbf{E}_h, p - p_h\|_h \leq c(h^{\alpha-1+\tau(1-\frac{\alpha}{\ell})} + \chi h^{1-\alpha})\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)},$$

which leads to (5.2) with $r := \min(1-\alpha, \alpha-1+\tau(1-\frac{\alpha}{\ell}))$ if $\nabla \cdot \mathbf{g} \neq 0$ and $r = \alpha-1+\tau(1-\frac{\alpha}{\ell})$ otherwise. Note that the assumed lower bound on α ensures that we have a convergence result as $h \rightarrow 0$. \square

Remark 5.1. Note that the best choice for α when $\nabla \cdot \mathbf{g} \neq 0$ is such that $1-\alpha = \alpha-1+\tau(1-\frac{\alpha}{\ell})$. This choice gives the following convergence rate $\frac{\tau}{2}(1-\frac{1}{\ell}) < r = \tau\frac{\ell-1}{2\ell-\tau} < \frac{\tau}{2}$. When $\nabla \cdot \mathbf{g} = 0$, the optimal choice consists of choosing $\alpha = 1$; the convergence rate is then $\tau(1-\frac{1}{\ell})$ and approaches τ as ℓ increases.

We now derive a convergence estimate assuming that the solution of (2.10) is smooth. In the next theorem we allow the parameter α to be any number in the interval $[0, 1]$.

Theorem 5.2. *Let $\mathbf{g} \in \mathbf{L}^2(\Omega)$ and let (\mathbf{E}, p) and (\mathbf{E}_h, p_h) be the solution of (2.10) and (4.14), respectively. Assume moreover that $\mathbf{E} \in \mathbf{H}^{k+1}(\Omega_\Sigma)$ and $p \in H^{k+\alpha}(\Omega_\Sigma)$ for some $0 < k \leq \ell-1$. Then there exists $c > 0$, uniform in h , such that*

$$(5.6) \quad \|\mathbf{E} - \mathbf{E}_h, p - p_h\|_h \leq c h^k (\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)} + \|p\|_{H^{k+\alpha}(\Omega_\Sigma)}).$$

Proof. The proof is similar to that of Theorem 5.1. We start from the triangular inequality

$$\|\mathbf{E} - \mathbf{E}_h, p - p_h\|_h \leq \|\mathbf{E} - \mathcal{C}_h^l \mathbf{E}, p - \mathcal{C}_h^p p\|_h + \|\mathcal{C}_h^l \mathbf{E} - \mathbf{E}_h, \mathcal{C}_h^p p - p_h\|_h,$$

We bound the two terms in the right hand side separately. For the first one, we use the local approximation properties of the operators \mathcal{C}_h^l and \mathcal{C}_h^p to derive

$$\begin{aligned} \|\mathbf{E} - \mathcal{C}_h^l \mathbf{E}, p - \mathcal{C}_h^p p\|_h &\leq c \left(h^k \|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)} + h^{k+\alpha} \|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)} + h^{-\frac{1}{2}} \|\mathbf{E} - \mathcal{C}_h^l \mathbf{E}\|_{\mathbf{L}^2(\Sigma \cup \Gamma)} \right. \\ &\quad \left. + h^{1-\alpha} h^{k+\alpha-1} \|p\|_{H^{k+\alpha}(\Omega_\Sigma)} + h^{\alpha-\frac{1}{2}} \|\mathbf{E} - \mathcal{C}_h^l \mathbf{E}\|_{\mathbf{L}^2(\Sigma)} \right). \end{aligned}$$

Using (A.4) for any $\sigma \in (\frac{1}{2}, 1)$, we have

$$\|\mathbf{E} - \mathcal{C}_h^l \mathbf{E}\|_{\mathbf{L}^2(\Sigma \cup \Gamma)} \leq c \|\mathbf{E} - \mathcal{C}_h^l \mathbf{E}\|_{\mathbf{L}^2(\Omega_\Sigma)}^{1-\frac{1}{2\sigma}} \|\mathbf{E} - \mathcal{C}_h^l \mathbf{E}\|_{\mathbf{H}^\sigma(\Omega_\Sigma)}^{\frac{1}{2\sigma}} \leq c h^{k+\frac{1}{2}} \|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)}.$$

As a result, we obtain

$$(5.7) \quad \|\mathbf{E} - \mathcal{C}_h^l \mathbf{E}, p - \mathcal{C}_h^p p\|_h \leq c h^k (\|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)} + \|p\|_{H^{k+\alpha}(\Omega_\Sigma)}).$$

Now we turn our attention to $\|\mathcal{C}_h^l \mathbf{E} - \mathbf{E}_h, \mathcal{C}_h^p p - p_h\|_h$. We use the coercivity of a_h , the Galerkin orthogonality and the continuity of a_h (for any $\sigma \in (0, \frac{1}{2})$) to get

$$\begin{aligned} \|\mathcal{C}_h^l \mathbf{E} - \mathbf{E}_h, \mathcal{C}_h^p p - p_h\|_h &\leq c (\|\mathbf{E} - \mathcal{C}_h^l \mathbf{E}, p - \mathcal{C}_h^p p\|_h + h^{\alpha-1} \|\mathbf{E} - \mathcal{C}_h^l \mathbf{E}\|_{\mathbf{L}^2(\Omega)} \\ &\quad + h^\sigma \|\kappa \nabla \times (\mathbf{E} - \mathcal{C}_h^l \mathbf{E})\|_{\mathbf{H}^\sigma(\mathcal{T}_h)} \\ &\quad + h \|\nabla \times \kappa \nabla \times (\mathbf{E} - \mathcal{C}_h^l \mathbf{E})\|_{\mathbf{L}^2(\mathcal{T}_h)} \\ &\quad + h^{-\alpha} \|p - \mathcal{C}_h^p p\|_{L^2(\Omega)} + h^{\frac{1}{2}-\alpha} \|p - \mathcal{C}_h^p p\|_{L^2(\Sigma)}). \end{aligned}$$

Using the approximation properties of \mathcal{C}_h^l together with (5.7), we infer

$$\begin{aligned} \|\mathbf{E} - \mathcal{C}_h^l \mathbf{E}, p - \mathcal{C}_h^p p\|_h &\leq ch^k (\|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)} + \|p\|_{H^{k+\alpha}(\Omega_\Sigma)}), \\ h^{\alpha-1} \|\mathbf{E} - \mathcal{C}_h^l \mathbf{E}\|_{\mathbf{L}^2(\Omega)} &\leq ch^{k+\alpha} \|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)}, \\ h^\sigma \|\kappa \nabla \times (\mathbf{E} - \mathcal{C}_h^l \mathbf{E})\|_{\mathbf{H}^\sigma(\mathcal{T}_h)} &\leq ch^k \|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)}, \\ h^{-\alpha} \|p - \mathcal{C}_h^p p\|_{L^2(\Omega)} &\leq ch^k \|p\|_{H^{k+\alpha}(\Omega_\Sigma)}. \end{aligned}$$

For the last term involving p , we use (A.4) for some $\sigma \in (\frac{1}{2}, 1)$:

$$\begin{aligned} h^{\frac{1}{2}-\alpha} \|p - \mathcal{C}_h^p p\|_{L^2(\Sigma)} &\leq ch^{\frac{1}{2}-\alpha} \|p - \mathcal{C}_h^p p\|_{L^2(\Omega_\Sigma)}^{1-\frac{1}{2\sigma}} \|p - \mathcal{C}_h^p p\|_{H^\sigma(\Omega_\Sigma)}^{\frac{1}{2\sigma}} \\ &\leq ch^{\frac{1}{2}-\alpha} h^{k+\alpha-\frac{1}{2}} \|p\|_{H^{k+\alpha}(\Omega_\Sigma)} = ch^k \|p\|_{H^{k+\alpha}(\Omega_\Sigma)}. \end{aligned}$$

For the last term involving \mathbf{E} , we distinguish two cases depending whether $k < 1$ or $k \geq 1$. If $k < 1$, we use an inverse inequality together with the approximation properties of \mathcal{C}_h^l to deduce that

$$\begin{aligned} h \|\nabla \times \kappa \nabla \times (\mathbf{E} - \mathcal{C}_h \mathbf{E})\|_{\mathbf{L}^2(\mathcal{T}_h)} &\leq h \|\nabla \times \kappa \nabla \times \mathbf{E}\|_{\mathbf{L}^2(\Omega)} + ch \|\mathcal{C}_h \mathbf{E}\|_{\mathbf{H}^2(\mathcal{T}_h)} \\ &\leq h \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} + h^k \|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)}. \end{aligned}$$

If $k \geq 1$, we use the local approximation properties of \mathcal{C}_h^l to get

$$h \|\nabla \times \kappa \nabla \times (\mathbf{E} - \mathcal{C}_h \mathbf{E})\|_{\mathbf{L}^2(\mathcal{T}_h)} \leq ch \|\mathbf{E} - \mathcal{C}_h \mathbf{E}\|_{\mathbf{H}^2(\mathcal{T}_h)} \leq ch^k \|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)}.$$

In both cases, we have:

$$h \|\nabla \times \kappa \nabla \times (\mathbf{E} - \mathcal{C}_h \mathbf{E})\|_{\mathbf{L}^2(\mathcal{T}_h)} \leq ch^k (\|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)} + \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}).$$

Gathering all the above estimates and using (5.7) gives the desired result (5.6). \square

Remark 5.2. Note that the error estimate (5.6) is optimal since it implies that $\|\nabla \times (\mathbf{E} - \mathbf{E}_h)\|_{\mathbf{L}^2(\Omega_\Sigma)} \leq ch^k$, which is the best that can be expected from piecewise polynomial approximation of degree k . Note also that there is no lower bound on α to get convergence when the solution of (2.10) is smooth, i.e., any α in the range $[0, 1]$ is acceptable.

5.2. Convergence in the \mathbf{L}^2 -norm. Before proving that the discrete solution converges to the exact solution in the \mathbf{L}^2 -norm, we prove a global version of Lemma A.3 that will be useful in the proof of Theorem 5.3.

Lemma 5.2. *Let $s \in (0, \frac{1}{2})$. Then there exists $c > 0$, uniform in h , such that the following holds, for any $\psi \in \mathbf{H}_{\text{curl}}(\Omega) \cap \mathbf{H}^s(\Omega)$ and any $\mathbf{F}_h \in \mathbf{X}_h$:*

$$\begin{aligned} (5.8) \quad |(\psi, [\mathbf{F}_h \times \mathbf{n}])_{\Sigma \cup \Gamma}| &\leq ch^{-\frac{1}{2}} \|[\mathbf{F}_h \times \mathbf{n}]\|_{\mathbf{L}^2(\Sigma \cup \Gamma)} (h^s \|\psi\|_{\mathbf{H}^s(\Omega)} + h \|\nabla \times \psi\|_{\mathbf{L}^2(\Omega)}). \end{aligned}$$

Proof. Let us consider $\psi \in \mathbf{H}_{\text{curl}}(\Omega) \cap \mathbf{H}^s(\Omega)$ and $\mathbf{F}_h \in \mathbf{X}_h$. Notice that the left hand side is well defined owing to Lemma A.3. We start from

$$|(\psi, [\mathbf{F}_h \times \mathbf{n}])_{\Sigma \cup \Gamma}| \leq \underbrace{|(\psi - \mathcal{K}_\delta \psi, [\mathbf{F}_h \times \mathbf{n}])_{\Sigma \cup \Gamma}|}_{:=I_1} + \underbrace{|(\mathcal{K}_\delta \psi, [\mathbf{F}_h \times \mathbf{n}])_{\Sigma \cup \Gamma}|}_{:=I_2},$$

for some δ to be defined later. We handle the two terms I_1, I_2 separately. For the first one, we apply Lemma A.3 with $\mathbf{v} = [\mathbf{F}_h \times \mathbf{n}]$, $\phi = \psi - \mathcal{K}_\delta \psi$ and $\sigma = s$, and we sum over all the faces $F \in \Sigma \cup \Gamma$. This leads to

$$\begin{aligned} I_1 &\leq c h^{-\frac{1}{2}} \|[\mathbf{F}_h \times \mathbf{n}]\|_{\mathbf{L}^2(\Sigma \cup \Gamma)} (h^s \|\psi - \mathcal{K}_\delta \psi\|_{\mathbf{H}^s(\Omega_\Sigma)} \\ &\quad + h \|\nabla \times (\psi - \mathcal{K}_\delta \psi)\|_{\mathbf{L}^2(\Omega_\Sigma)} + \|\psi - \mathcal{K}_\delta \psi\|_{\mathbf{L}^2(\Omega_\Sigma)}) \\ &\leq c h^{-\frac{1}{2}} \|[\mathbf{F}_h \times \mathbf{n}]\|_{\mathbf{L}^2(\Sigma \cup \Gamma)} (h^s \|\psi - \mathcal{K}_\delta \psi\|_{\mathbf{H}^s(\Omega_\Sigma)} \\ &\quad + h \|\nabla \times \psi\|_{\mathbf{L}^2(\Omega_\Sigma)} + h \|\nabla \times \mathcal{K}_\delta \psi\|_{\mathbf{L}^2(\Omega_\Sigma)} + \|\psi - \mathcal{K}_\delta \psi\|_{\mathbf{L}^2(\Omega_\Sigma)}). \end{aligned}$$

Using the approximation properties of \mathcal{K}_δ (3.16) and (3.18), we arrive at

$$\begin{aligned} I_1 &\leq c h^{-\frac{1}{2}} \|[\mathbf{F}_h \times \mathbf{n}]\|_{\mathbf{L}^2(\Sigma \cup \Gamma)} (h^s \|\psi\|_{\mathbf{H}^s(\Omega_\Sigma)} \\ &\quad + h \|\nabla \times \psi\|_{\mathbf{L}^2(\Omega_\Sigma)} + \delta^s \|\psi\|_{\mathbf{H}^s(\Omega_\Sigma)} + h \|\mathcal{K}_\delta \psi\|_{\mathbf{H}^1(\Omega_\Sigma)}) \\ &\leq c h^{-\frac{1}{2}} \|[\mathbf{F}_h \times \mathbf{n}]\|_{\mathbf{L}^2(\Sigma \cup \Gamma)} ((h^s + \delta^s + h \delta^{s-1}) \|\psi\|_{\mathbf{H}^s(\Omega_\Sigma)} + h \|\nabla \times \psi\|_{\mathbf{L}^2(\Omega_\Sigma)}). \end{aligned}$$

We handle I_2 by using the Cauchy-Schwarz inequality on every $\partial\Omega_i$, $i = 1, \dots, N$.

$$I_2 \leq c h^{-\frac{1}{2}} \|[\mathbf{F}_h \times \mathbf{n}]\|_{\mathbf{L}^2(\Sigma \cup \Gamma)} \sum_{i=1}^N h^{\frac{1}{2}} \|\mathcal{K}_\delta \psi\|_{\mathbf{L}^2(\partial\Omega_i)}.$$

We use (A.5) on every Ω_i with $\Theta := \frac{1-2s}{2(1-s)}$ to obtain

$$\begin{aligned} I_2 &\leq c h^{-\frac{1}{2}} \|[\mathbf{F}_h \times \mathbf{n}]\|_{\mathbf{L}^2(\Sigma \cup \Gamma)} \sum_{i=1}^N h^{\frac{1}{2}} \|\mathcal{K}_\delta \psi\|_{\mathbf{H}^s(\Omega_i)}^{1-\Theta} \|\mathcal{K}_\delta \psi\|_{\mathbf{H}^1(\Omega_i)}^\Theta \\ &\leq c h^{-\frac{1}{2}} \|[\mathbf{F}_h \times \mathbf{n}]\|_{\mathbf{L}^2(\Sigma \cup \Gamma)} h^{\frac{1}{2}} \|\mathcal{K}_\delta \psi\|_{\mathbf{H}^s(\Omega_\Sigma)}^{1-\Theta} \|\mathcal{K}_\delta \psi\|_{\mathbf{H}^1(\Omega_\Sigma)}^\Theta, \end{aligned}$$

where the constant c depends on N , which we recall is a fixed number. Using again the approximation properties of \mathcal{K}_δ we infer that

$$\begin{aligned} I_2 &\leq c h^{-\frac{1}{2}} \|[\mathbf{F}_h \times \mathbf{n}]\|_{\mathbf{L}^2(\Sigma \cup \Gamma)} h^{\frac{1}{2}} \delta^{(s-1)\Theta} \|\psi\|_{\mathbf{H}^s(\Omega_\Sigma)} \\ &\leq c h^{-\frac{1}{2}} \|[\mathbf{F}_h \times \mathbf{n}]\|_{\mathbf{L}^2(\Sigma \cup \Gamma)} h^{\frac{1}{2}} \delta^{s-\frac{1}{2}} \|\psi\|_{\mathbf{H}^s(\Omega_\Sigma)}. \end{aligned}$$

Then (5.8) is obtained by gathering the above estimates and setting $\delta = h$. \square

Remark 5.3 (Alternative Decomposition). Estimate (5.8) can alternatively be derived using the decomposition $\psi = \psi - \mathcal{C}_h^l \psi + \mathcal{C}_h^l \psi$ instead of $\psi = \psi - \mathcal{K}_\delta \psi + \mathcal{K}_\delta \psi$.

Theorem 5.3. *Let $\mathbf{g} \in \mathbf{L}^2(\Omega)$ and let (\mathbf{E}, p) be the solution of (2.10). Let $\tau < \min(\tau_\varepsilon, \tau_\kappa)$ where τ_ε and τ_κ are defined in Theorem 2.1. Let (\mathbf{E}_h, p_h) be solution of (4.14). For any $\alpha \in \left(\frac{\ell(1-\tau)}{\ell-\tau}, 1\right)$, there exists $c > 0$, uniform in h , such that*

$$(5.9) \quad \|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{L}^2(\Omega)} \leq c h^{2r} \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)},$$

with $r := \min(1 - \alpha, \alpha - 1 + \tau(1 - \frac{\alpha}{\ell}))$. If in addition $\mathbf{E} \in \mathbf{H}^{k+1}(\Omega_\Sigma)$ and $p \in H^{k+\alpha}(\Omega_\Sigma)$ for some $0 < k < \ell - 1$, then the following holds:

$$(5.10) \quad \|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{L}^2(\Omega)} \leq c h^{k+r} (\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)} + \|p\|_{H^{k+\alpha}(\Omega_\Sigma)}).$$

Proof. We are going to use a duality argument à la Nitsche-Aubin. In the following we denote a_h^1 the extension to $[(\mathbf{Z}^\tau(\Omega) + \mathbf{X}_h) \times H_0^1(\Omega)]^2$ of the bilinear form defined on $[\mathbf{X}_h \times M_h]^2$ in (4.13) with $\theta = 1$. Then the following symmetry property holds:

$$a_h^1((\mathbf{F}, q), (\mathbf{G}, r)) = a_h^1((\mathbf{G}, -r), (\mathbf{F}, -q)).$$

for all $((\mathbf{F}, q), (\mathbf{G}, r)) \in [(\mathbf{Z}^\tau(\Omega) + \mathbf{X}_h) \times H_0^1(\Omega)]^2$. Let $(\mathbf{w}, q) \in \mathbf{H}_{0,\text{curl}}(\Omega) \times H_0^1(\Omega)$ be the solution of the following (adjoint) problem:

$$\nabla \times (\kappa \nabla \times \mathbf{w}) - \varepsilon \nabla q = \varepsilon (\mathbf{E} - \mathbf{E}_h), \quad \nabla \cdot (\varepsilon \mathbf{w}) = 0.$$

Recall that Theorem 2.1 implies that $\mathbf{w} \in \mathbf{Z}^\tau(\Omega) \cap \mathbf{H}^\tau(\Omega)$ for any $s < \tau$ and that

$$(5.11) \quad \|\mathbf{w}\|_{\mathbf{H}^\tau(\Omega)} + \|\kappa \nabla \times \mathbf{w}\|_{\mathbf{H}^\tau(\Omega)} + \|\nabla \times \kappa \nabla \times \mathbf{w}\|_{\mathbf{L}^2(\Omega)} \leq c \|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{L}^2(\Omega)}.$$

The definition of the pair (\mathbf{w}, q) implies that $(\varepsilon \nabla q, \nabla \varphi)_\Omega = -(\varepsilon (\mathbf{E} - \mathbf{E}_h), \nabla \varphi)$ for all $\varphi \in H_0^1(\Omega)$, and the following identities hold:

$$\begin{aligned} \|\varepsilon^{\frac{1}{2}}(\mathbf{E} - \mathbf{E}_h)\|_{\mathbf{L}^2(\Omega)}^2 &= a_h^1((\mathbf{w}, -q), (\mathbf{E} - \mathbf{E}_h, p_h - p)) + c_\alpha h^{2(1-\alpha)} (\varepsilon \nabla q, \nabla(p_h - p))_\Omega \\ &= a_h^1((\mathbf{E} - \mathbf{E}_h, p - p_h), (\mathbf{w}, q)) + c_\alpha h^{2(1-\alpha)} (\varepsilon (\mathbf{E} - \mathbf{E}_h), \nabla(p - p_h))_\Omega \\ &= a_h((\mathbf{E} - \mathbf{E}_h, p - p_h), (\mathbf{w}, q)) + c_\alpha h^{2(1-\alpha)} (\varepsilon (\mathbf{E} - \mathbf{E}_h), \nabla(p - p_h))_\Omega \\ &\quad + (1 - \theta) (\llbracket \kappa \nabla \times \mathbf{w} \rrbracket, \llbracket -\mathbf{E}_h \times \mathbf{n} \rrbracket)_{\Sigma \cup \Gamma} \end{aligned}$$

We now use the Galerkin orthogonality and we introduce the global approximation $\mathcal{C}_h^g \mathcal{K}_\delta \mathbf{w}$, with $\delta = h^{1-\frac{\alpha}{2}}$, and the pressure approximation $\mathcal{C}_h^p q$:

$$(5.12) \quad \|\varepsilon^{\frac{1}{2}}(\mathbf{E} - \mathbf{E}_h)\|_{\mathbf{L}^2(\Omega)}^2 = a_h((\mathbf{E} - \mathbf{E}_h, p - p_h), (\mathbf{w} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{w}, q - \mathcal{C}_h^p q)) + c_\alpha h^{2(1-\alpha)} (\varepsilon (\mathbf{E} - \mathbf{E}_h), \nabla(p - p_h))_\Omega - (1 - \theta) (\kappa \nabla \times \mathbf{w}, \llbracket \mathbf{E}_h \times \mathbf{n} \rrbracket)_{\Sigma \cup \Gamma}.$$

Note that we replaced $\llbracket \kappa \nabla \times \mathbf{w} \rrbracket$ by $\kappa \nabla \times \mathbf{w}$ since the tangent component of $\kappa \nabla \times \mathbf{w}$ is continuous across the interfaces owing to $\nabla \times (\kappa \nabla \times \mathbf{w}) \in \mathbf{L}^2(\Omega)$.

We now handle the three terms in the right hand side separately. For the first one, we use Proposition 4.4 with $s = 1 - \alpha$, $\mathbf{F} = \mathbf{w}$ and $\mathbf{F}_h = \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{w}$ (note that $\mathbf{F}_h \in \mathbf{Y}_h \subset \mathbf{X}_h \cap \mathbf{H}_{0,\text{curl}}(\Omega)$); we then infer that

$$\begin{aligned} |a_h((\mathbf{E} - \mathbf{E}_h, p - p_h), (\mathbf{w} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{w}, q - \mathcal{C}_h^p q))| &\leq \\ &c \|\mathbf{E} - \mathbf{E}_h, p - p_h\|_h (\|\mathbf{w} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{w}, q - \mathcal{C}_h^p q\|_h \\ &+ h^{\alpha-1} \|\mathbf{w} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{w}\|_{\mathbf{L}^2(\Omega)} + h^{-\alpha} \|q - \mathcal{C}_h^p q\|_{L^2(\Omega)} + h^{\frac{1}{2}-\alpha} \|q - \mathcal{C}_h^p q\|_{L^2(\Sigma)} \\ &+ h \|\nabla \times \kappa \nabla \times (\mathbf{w} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{w})\|_{\mathbf{L}^2(\mathcal{T}_h)} + h^{1-\alpha} \|\nabla \times (\mathbf{w} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{w})\|_{\mathbf{H}^{1-\alpha}(\Omega)}). \end{aligned}$$

The right hand side has already been estimated in the proof of Theorem 5.1. We then have

$$(5.13) \quad \begin{aligned} &|a_h((\mathbf{E} - \mathbf{E}_h, p - p_h), (\mathbf{w} - \mathcal{C}_h^g \mathcal{K}_\delta \mathbf{w}, \mathcal{C}_h^p q - q))| \\ &\leq c \|\mathbf{E} - \mathbf{E}_h, p - p_h\|_h h^r \|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

The second term in (5.12) is estimated by using the Cauchy-Schwarz inequality, the definition of the norm $\|\cdot\|_h$ and the inequality $r \leq 1 - \alpha$,

$$(5.14) \quad \begin{aligned} &\left| h^{2(1-\alpha)} (\varepsilon (\mathbf{E} - \mathbf{E}_h), \nabla(p - p_h))_\Omega \right| \leq c h^{1-\alpha} \|\nabla(p - p_h)\|_{\mathbf{L}^2(\Omega)} h^{1-\alpha} \|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{L}^2(\Omega)} \\ &\leq c \|\mathbf{E} - \mathbf{E}_h, p - p_h\|_h h^r \|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

The last term in (5.12) is estimated by using Lemma 5.2 with $\boldsymbol{\psi} := \kappa \nabla \times \mathbf{w}$ and $s := \tau$:

$$(5.15) \quad \begin{aligned} &|(1 - \theta) (\kappa \nabla \times \mathbf{w}, \llbracket \mathbf{E}_h \times \mathbf{n} \rrbracket)_{\Sigma \cup \Gamma}| \\ &\leq c \|\mathbf{E} - \mathbf{E}_h\|_h (h^\tau \|\kappa \nabla \times \mathbf{w}\|_{\mathbf{H}^\tau(\Omega)} + h \|\nabla \times (\kappa \nabla \times \mathbf{w})\|_{\mathbf{L}^2(\Omega)}) \\ &\leq c \|\mathbf{E} - \mathbf{E}_h\|_h h^r \|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

where we have used (5.11) and $r \leq \frac{\tau}{2} < \tau$. Upon inserting (5.13)-(5.14)-(5.15) in (5.12) we obtain

$$\|\varepsilon^{\frac{1}{2}}(\mathbf{E} - \mathbf{E}_h)\|_{\mathbf{L}^2(\Omega)}^2 \leq ch^r \|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{L}^2(\Omega)} \|\mathbf{E} - \mathbf{E}_h, p - p_h\|_h.$$

Owing to the uniform positivity of ε , this leads to:

$$\|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{L}^2(\Omega)} \leq ch^r \|\mathbf{E} - \mathbf{E}_h, p - p_h\|_h.$$

Now we consider two cases. Assuming only minimal regularity, Theorem 5.1 gives a bound on $\|\mathbf{E} - \mathbf{E}_h, p - p_h\|_h$ that leads to (5.9). If \mathbf{E} and p are piecewise smooth, then we can apply Theorem 5.2 and we obtain (5.10). \square

Remark 5.4. Let $\tau \in (0, \frac{1}{2})$ and denote (\mathbf{E}, p) the solution of (2.10). Assume that $\mathbf{E} \in \mathbf{H}^r(\Omega)$ and $\mathbf{E} \notin \mathbf{H}^{\tau^+}(\Omega)$ for all $\tau^+ > \tau$. Then the best choice of α is $\alpha = \frac{\ell(2-\tau)}{2\ell-\tau}$, which gives the convergence rate $2r = \tau \frac{\ell-1}{\ell-\frac{1}{2}}$; this convergence rate approaches the optimal rate, τ , when the approximation degree ℓ is large.

Remark 5.5. Note that the degree of the polynomials used for M_h is not involved in the convergence rate when minimal regularity is assumed. This means that we can use different degrees of polynomials for \mathbf{X}_h and M_h , and that it is sufficient to take polynomials of degree 1 for M_h to get convergence.

5.3. Numerical illustrations. In this section we illustrate numerically the performance of the method on a boundary value problem on the L -shaped domain

$$\Omega = (-1, 1)^2 \setminus ([0, +1] \times [-1, 0]).$$

We assume that Ω is composed of three subdomains:

$$\Omega_1 = (0, 1)^2, \quad \Omega_2 = (-1, 0) \times (0, 1), \quad \Omega_3 = (-1, 0)^2.$$

We use $\kappa \equiv 1$ in Ω , $\varepsilon|_{\Omega_2} = 1$ and $\varepsilon|_{\Omega_1} = \varepsilon|_{\Omega_3} =: \varepsilon_r$. Denoting $\lambda > 0$ a real number such that $\tan\left(\frac{\lambda\pi}{4}\right) \tan\left(\frac{\lambda\pi}{2}\right) = \varepsilon_r$, we define the scalar potential $S_\lambda(r, \theta) = r^\lambda \phi_\lambda(\theta)$, where (r, θ) are the polar coordinates, and ϕ_λ is defined by

$$\phi_\lambda(\theta) = \begin{cases} \sin(\lambda\theta) & \text{if } 0 \leq \theta < \frac{\pi}{2}, \\ \frac{\sin(\frac{\pi}{2})}{\cos(\frac{\lambda}{4}\pi)} \cos\left(\lambda\left(\theta - \frac{3}{4}\pi\right)\right) & \text{if } \frac{\pi}{2} \leq \theta < \pi, \\ \sin\left(\lambda\left(\frac{3}{2}\pi - \theta\right)\right) & \text{if } \pi \leq \theta \leq \frac{3\pi}{2}. \end{cases}$$

Then we solve the problem

$$(5.16) \quad \nabla \times \nabla \times \mathbf{E} = 0, \quad \nabla \cdot (\varepsilon \mathbf{E}) = 0, \quad \mathbf{E} \times \mathbf{n}|_{\partial\Omega} = \nabla S_\lambda \times \mathbf{n}.$$

The exact solution is $\mathbf{E} = \nabla S_\lambda$. We present two series of simulations in Table 1. We use $\lambda = 0.535$ in Table 1(a) and $\lambda = 0.24$ in Table 1(b), which gives $\varepsilon_r = 0.499 \pm 10^{-3}$ and $\varepsilon_r \simeq 7.55 \cdot 10^{-2}$, respectively. The relative error in the \mathbf{L}^2 -norm is reported in the column “rel. err.” and the convergence rate is reported in the column “coc”. Several values of α are used to evaluate the effect of λ and α on the convergence rates.

We observe that the convergence rate is quasi-optimal when α is close to 1, which is consistent with Remark 5.1, since (5.16) can be re-written in the form (2.10) with $\nabla \cdot \mathbf{g} = 0$.

TABLE 1. \mathbf{L}^2 -errors and convergence rates. The convergence rates are almost optimal for $\alpha = 0.9$ in both cases.

(a) \mathbf{L}^2 -errors and convergence rates for $\lambda = 0.535$

h	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.9$	
	rel. err.	coc	rel. err.	coc	rel. err.	coc
0.2	2.332E-1	-	1.444E-1	-	1.249E-1	-
0.1	2.473E-1	-0.08	1.168E-1	0.31	8.846E-2	0.50
0.05	2.631E-1	-0.09	9.452E-2	0.31	6.186E-2	0.52
0.025	2.797E-1	-0.09	7.700E-2	0.30	4.289E-2	0.53
0.0125	2.968E-1	-0.09	6.312E-2	0.29	2.962E-2	0.53

(b) \mathbf{L}^2 -errors and convergence rates for $\lambda = 0.24$

h	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.9$	
	rel. err.	coc	rel. err.	coc	rel. err.	coc
0.2	5.773E-1	-	4.739E-1	-	4.426E-1	-
0.1	6.209E-1	-0.11	4.507E-1	0.07	3.801E-1	0.22
0.05	6.711E-1	-0.11	4.413E-1	0.03	3.259E-1	0.22
0.025	7.180E-1	-0.10	4.452E-1	-0.01	2.788E-1	0.23
0.0125	7.564E-1	-0.08	4.602E-1	-0.05	2.380E-1	0.23

6. EIGENVALUE PROBLEM

We extend in this section the theory introduced above to eigenvalue problems. We want to establish an approximation result for the solutions to the following problem: Find $(\mathbf{E}, \lambda) \in [\mathbf{H}_{0,\text{curl}}(\Omega) \cap \mathbf{H}_{\text{div}}(\Omega, \epsilon)] \times \mathbb{R}$ such that

$$(6.1) \quad \nabla \times \kappa \nabla \times \mathbf{E} = \lambda \epsilon \mathbf{E}.$$

We restrict ourselves in the rest of this section to the symmetric variant of the bilinear form a_h defined in (4.13), i.e., we set $\theta = 1$. We finally assume from now on that α is chosen as in Theorem 5.1, i.e., $\alpha \in \left(\frac{\ell(1-\tau)}{\ell-\tau}, 1\right)$, where τ is the minimal regularity index of the problem (2.10) as defined in Theorem 2.1. In the following we set $r := \min(1 - \alpha, \alpha - 1 + \tau(1 - \frac{\alpha}{\ell}))$.

6.1. Framework. Let us equip $\mathbf{L}^2(\Omega)$ with the inner product $(\mathbf{f}, \mathbf{g})_\epsilon := \int_\Omega \epsilon \mathbf{f} \cdot \mathbf{g}$. This inner product is equivalent to the usual \mathbf{L}^2 -inner product owing to (2.13). The associated norm is denoted $\|\cdot\|_\epsilon$.

For any $\mathbf{g} \in \mathbf{L}^2(\Omega)$, we denote (\mathbf{E}, p) the solution of (2.10) and we set $\mathbf{A}\mathbf{g} := \mathbf{E}$. This defines an operator $A : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$ that is self-adjoint and compact (cf. Theorem 2.1). We now define two families of discrete operators $\mathcal{E}_h : \mathbf{L}^2(\Omega) \rightarrow \mathbf{X}_h$ and $\mathcal{P}_h : \mathbf{L}^2(\Omega) \rightarrow M_h$ so that for any $\mathbf{g} \in \mathbf{L}^2(\Omega)$, the pair $(\mathcal{E}_h \mathbf{g}, \mathcal{P}_h \mathbf{g})$ solves (4.14). Then we finally define

$$(6.2) \quad \begin{aligned} A_h : \mathbf{L}^2(\Omega) &\longrightarrow \mathbf{X}_h + \nabla M_h \subset \mathbf{L}^2(\Omega) \\ \mathbf{g} &\longmapsto \mathcal{E}_h \mathbf{g} - c_\alpha h^{2(1-\alpha)} \nabla \mathcal{P}_h \mathbf{g}. \end{aligned}$$

We want to study the spectrally correctness of the family $\{A_h\}$ in the following sense:

Theorem 6.1 (Spectral correctness Babuška and Osborn [2], Osborn [21]). *Let X be an Hilbert space and $A : X \rightarrow X$ be a self-adjoint compact operator. Let $\Theta = \{h_n; n \in \mathbb{N}\}$ be a discrete subset of \mathbb{R} such that $h_n \rightarrow 0$ as $n \rightarrow +\infty$. Assume that there exists a family of operators $A_h : X \rightarrow X$, $h \in \Theta$, such that:*

- A_h is a linear self-adjoint operator, for all $h \in \Theta$.
- A_h converges pointwise to A .
- The family is collectively compact.

Let μ be an eigenvalue of A of multiplicity m and let $\{\phi_j\}$, $j = 1, \dots, m$ be a set of associated orthonormal eigenvectors.

- (i) For any $\epsilon > 0$ such that the disk $B(\mu, \epsilon)$ contains no other eigenvalues of A , there exists h_ϵ such that, for all $h < h_\epsilon$, A_h has exactly m eigenvalues (repeated according to their multiplicity) in the disk $B(\mu, \epsilon)$.
- (ii) In addition, for $h < h_\epsilon$, if we denote $\mu_{h,j}$, $j = 1, \dots, m$ the set of the eigenvalues of A_h in $B(\mu, \epsilon)$, there exists $c > 0$ such that

$$(6.3) \quad c|\mu - \mu_{h,j}| \leq \sum_{j,l=1}^m |((A - A_h)\phi_j, \phi_l)_X| + \sum_{j=1}^m \|(A - A_h)\phi_j\|_X^2.$$

6.2. Approximation result. We start by proving that the operators $\{A_h\}$ are self-adjoint, then we prove the pointwise convergence, and we finally establish the collective compactness.

Lemma 6.1. *For any h , $A_h : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$ is a self-adjoint operator, i.e., for any $\mathbf{e}, \mathbf{f} \in \mathbf{L}^2(\Omega)$, the following holds*

$$(6.4) \quad (A_h \mathbf{e}, \mathbf{f})_\epsilon = (\mathbf{e}, A_h \mathbf{f})_\epsilon.$$

Proof. Let $\mathbf{e}, \mathbf{f} \in \mathbf{L}^2(\Omega)$. By definition we have

$$a_h((\mathcal{E}_h \mathbf{e}, \mathcal{P}_h \mathbf{e}), (\mathcal{E}_h \mathbf{f}, -\mathcal{P}_h \mathbf{f})) = (\mathbf{e}, \mathcal{E}_h \mathbf{f})_\epsilon - c_\alpha h^{2(1-\alpha)} (\mathbf{e}, \nabla \mathcal{P}_h \mathbf{f})_\epsilon = (\mathbf{e}, A_h \mathbf{f})_\epsilon.$$

Using the symmetry properties of a_h , we infer

$$\begin{aligned} a_h((\mathcal{E}_h \mathbf{e}, \mathcal{P}_h \mathbf{e}), (\mathcal{E}_h \mathbf{f}, -\mathcal{P}_h \mathbf{f})) &= a_h((\mathcal{E}_h \mathbf{f}, \mathcal{P}_h \mathbf{f}), (\mathcal{E}_h \mathbf{e}, -\mathcal{P}_h \mathbf{e})) \\ &= (\mathbf{f}, \mathcal{E}_h \mathbf{e})_\epsilon - c_\alpha h^{2(1-\alpha)} (\mathbf{f}, \nabla \mathcal{P}_h \mathbf{e})_\epsilon = (\mathbf{f}, A_h \mathbf{e})_\epsilon, \end{aligned}$$

thereby proving that the operator A_h is self-adjoint on the Hilbert space $\mathbf{L}^2(\Omega)$ equipped with the inner product $(\cdot, \cdot)_\epsilon$. \square

Lemma 6.2. *Under the above assumptions, there exists $c > 0$, uniform with respect to h such that,*

$$(6.5) \quad \forall \mathbf{e} \in \mathbf{L}^2(\Omega), \quad \|A_h \mathbf{e} - A \mathbf{e}\|_\epsilon \leq c h^{2r} \|\mathbf{e}\|_\epsilon.$$

Proof. Let $A \mathbf{e} \in \mathbf{L}^2(\Omega)$ and $p \in H_0^1(\Omega)$ such that $\nabla \times (\kappa \nabla \times A \mathbf{e}) + \epsilon \nabla p = \epsilon A \mathbf{e}$. Using the triangular inequality, Theorems 5.1 and 5.3, the equivalence between the norms on $\mathbf{L}^2(\Omega)$ and the fact that $r \leq 1 - \alpha$, we infer that

$$\begin{aligned} \|A \mathbf{e} - A_h \mathbf{e}\|_\epsilon &\leq \|A \mathbf{e} - \mathcal{E}_h A \mathbf{e}\|_\epsilon + c_\alpha h^{2(1-\alpha)} \|\nabla \mathcal{P}_h A \mathbf{e} - \nabla p\|_\epsilon + h^{2(1-\alpha)} \|\nabla p\|_\epsilon \\ &\leq c(h^{2r} \|\mathbf{e}\|_\epsilon + h^{1-\alpha} \|A \mathbf{e} - \mathcal{E}_h A \mathbf{e}\|_\epsilon + h^{2(1-\alpha)} \|\mathbf{e}\|_\epsilon) \leq c h^{2r} \|\mathbf{e}\|_\epsilon, \end{aligned}$$

which concludes the proof. \square

Note that the above result is stronger than the pointwise convergence hypothesis, i.e., A_h converges in norm to A . Now let us turn our attention to the question of collective compactness. Recall that a set $\mathcal{A} := \{A_h \in \mathcal{L}(X; X), h \in \Theta\}$ is said to be collectively compact if, for each bounded set $U \subset X$, the image set $\mathcal{A}U := \{A_h \mathbf{g}, \mathbf{g} \in U, A_h \in \mathcal{A}\}$ is relatively compact in X .

Lemma 6.3. *The family $\{A_h\}_{h>0}$ is collectively compact under the above assumptions.*

Proof. Owing to the compact embedding $\mathbf{H}^s(\Omega) \subset \mathbf{L}^2(\Omega)$ for any $s > 0$, it is sufficient to prove that there exists $s > 0$ and $c > 0$ such that, for any $\mathbf{g} \in \mathbf{L}^2(\Omega)$ and any $h > 0$,

$$\|A_h \mathbf{g}\|_{\mathbf{H}^s(\Omega)} \leq c \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}.$$

Let us take $\mathbf{g} \in \mathbf{L}^2(\Omega)$. Owing to the definition of \mathbf{X}_h and M_h , we know that $A_h \mathbf{g} \in \mathbf{H}^s(\Omega)$ for any $s \in (0, \frac{1}{2})$. Moreover, there exists c , only depending on s and the shape regularity of the mesh family, such that the following inverse inequality holds:

$$\|A_h \mathbf{g}\|_{\mathbf{H}^s(\Omega)} \leq c h^{-s} \|A_h \mathbf{g}\|_{\mathbf{L}^2(\Omega)}.$$

Let us consider $s < r$ and notice that $2s < \tau$. Using the triangular inequality, interpolation results, the above inverse inequality together with Theorems 5.3 and 2.1 leads to:

$$\begin{aligned} \|A_h \mathbf{g}\|_{\mathbf{H}^s(\Omega)} &\leq \|A_h \mathbf{g} - A \mathbf{g}\|_{\mathbf{H}^s(\Omega)} + \|A \mathbf{g}\|_{\mathbf{H}^s(\Omega)} \\ &\leq c \|A_h \mathbf{g} - A \mathbf{g}\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{2}} \|A_h \mathbf{g} - A \mathbf{g}\|_{\mathbf{H}^{2s}(\Omega)}^{\frac{1}{2}} + c \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} \\ &\leq c h^r \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{2}} \left(h^{-s} \|A_h \mathbf{g}\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{2}} + \|A \mathbf{g}\|_{\mathbf{H}^{2s}(\Omega)}^{\frac{1}{2}} \right) + c \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} \\ &\leq c (h^{r-s} + 1) \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

This implies the collective compactness of $\{A_h\}$ since $r > s$. \square

We conclude that the approximation is spectrally correct in the sense of Theorem 6.1 by combining Lemmas 6.1, 6.2, 6.3.

6.3. Numerical illustration. In this section, we present some eigenvalues computations. We consider the square $\Omega = (-1, 1)^2$ in the plane. We divide Ω into four subdomains

$$\Omega_1 = (0, 1)^2, \quad \Omega_2 = (-1, 0) \times (0, 1), \quad \Omega_3 = (-1, 0)^2, \quad \Omega_4 = (0, 1) \times (-1, 0).$$

We use $\kappa \equiv 1$ in Ω , $\varepsilon|_{\Omega_1} = \varepsilon|_{\Omega_3} = 1$ and $\varepsilon|_{\Omega_2} = \varepsilon|_{\Omega_4} = \varepsilon_r$. A benchmark results are provided for $\varepsilon_r^{-1} \in \{2, 10, 100, 10^8\}$ in Dauge [13]. Tables 2 and 3 show results for $\varepsilon_r = 0.5$ and $\varepsilon_r = 0.1$ respectively. The ratio $\frac{|\lambda_c - \lambda_r|}{\lambda_r}$ is reported in column “rel. err.”, where λ_c and λ_r are the computed and reference eigenvalues, respectively. The reference values are those from the benchmark. The computed order of convergence is shown in the column “coc”. The computations have been done using ARPACK (cf. Lehoucq et al. [18]) with tolerance 10^{-8} .

Note that the computed order of convergence seems to reach a constant value for sufficiently small h , for every eigenvalue, as expected.

TABLE 2. Approximation of the first four eigenvalues for $\varepsilon_r = 0.5$.
We used $\alpha = 0.7$ in the simulations.

h	$\lambda_r \simeq 3.3175$		$\lambda_r \simeq 3.3663$		$\lambda_r \simeq 6.1863$		$\lambda_r \simeq 13.926$	
	rel. err.	coc	rel. err.	coc	rel. err.	coc	rel. err.	coc
0.2	9.364E-4	-	3.943E-3	-	1.439E-1	-	6.104E-1	-
0.1	1.833E-4	2.35	2.147E-3	0.88	1.734E-4	9.70	4.484E-1	0.44
0.05	3.751E-5	2.29	1.188E-3	0.85	2.241E-5	2.95	1.599E-1	1.49
0.025	8.405E-6	2.16	6.463E-4	0.88	2.833E-6	2.98	1.120E-5	13.8
0.0125	2.081E-6	2.01	3.439E-4	0.91	3.667E-7	2.95	1.478E-6	2.92

TABLE 3. Approximation of the first four eigenvalues for $\varepsilon_r = 0.1$.
We used $\alpha = 0.8$ in the simulations.

h	$\lambda_r \simeq 4.5339$		$\lambda_r \simeq 6.2503$		$\lambda_r \simeq 7.0371$		$\lambda_r \simeq 22.342$	
	rel. err.	coc	rel. err.	coc	rel. err.	coc	rel. err.	coc
0.2	4.559E-1	-	6.052E-1	-	6.410E-1	-	8.869E-1	-
0.1	2.859E-1	0.67	4.731E-1	0.36	5.310E-1	0.27	8.512E-1	0.06
0.05	3.306E-2	3.11	2.982E-1	0.67	3.763E-1	0.50	8.033E-1	0.08
0.025	2.154E-6	13.9	7.748E-2	1.94	1.772E-1	1.09	7.406E-1	0.12
0.0125	2.608E-7	3.05	3.258E-3	4.57	5.946E-7	18.2	6.602E-1	0.17

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APPENDIX A. TECHNICAL LEMMAS

Let $\{\mathcal{T}_h\}_{h>0}$ be an affine shape-regular mesh family in \mathbb{R}^3 . Let $T_K : \hat{K} \rightarrow K$ be the affine mapping that maps the reference element \hat{K} to K and let J_K be the Jacobian of T_K . It is a standard result that there are constants that depend only on \hat{K} and the shape regularity constants of the mesh family so that

$$(A.1) \quad \|J_K\| \leq ch_K, \quad \|J_K^{-1}\| \leq ch_K^{-1}, \quad |\det(J_K)| \leq ch_K^d, \quad |\det(J_K^{-1})| \leq ch_K^{-d},$$

where h_K is the diameter of K .

Lemma A.1. *For all $s \in [0, 1]$, there is a constant c , uniform with respect to the mesh family, so that the following holds for all cells $K \in \mathcal{T}_h$ and all $\psi \in H^s(K)$ with zero average over K :*

$$(A.2) \quad \|\hat{\psi}\|_{H^s(\hat{K})} \leq ch_K^{s-\frac{d}{2}} \|\psi\|_{H^s(K)}, \quad \text{where} \quad \hat{\psi}(\mathbf{x}) := \psi(T_K(\mathbf{x}))$$

Proof. Upon making the change of variable $\mathbf{x} = T_K(\hat{\mathbf{x}})$ we obtain

$$\|\hat{\psi}\|_{L^2(\hat{K})} = |\det(J_K)|^{-\frac{1}{2}} \|\psi\|_{L^2(K)} \leq ch_K^{-\frac{d}{2}} \|\psi\|_{L^2(K)}.$$

Likewise, using the fact that $\hat{\psi}$ is of zero average, the Poincaré inequality implies

$$\begin{aligned} \|\hat{\psi}\|_{H^1(\hat{K})} &= \left(\|\hat{\psi}\|_{L^2(\hat{K})}^2 + \|\nabla \hat{\psi}\|_{L^2(\hat{K})}^2 \right)^{\frac{1}{2}} \leq (c_p(\hat{K}) + 1)^{\frac{1}{2}} \|\hat{\nabla} \hat{\psi}\|_{H^1(\hat{K})} \\ &\leq c |\det(J_K)|^{-\frac{1}{2}} \|J_K\| \|\nabla \psi\|_{L^2(K)} \leq ch_K^{-\frac{d}{2}+1} \|\psi\|_{H^1(K)}. \end{aligned}$$

Then, the Riesz-Thorin theorem implies that

$$\|\widehat{\psi}\|_{\dot{H}^s(\widehat{K})} \leq c h_K^{s-\frac{3}{2}} \|\psi\|_{\dot{H}^s(K)},$$

where we defined $\dot{H}^s(E) := [\dot{L}^2(E), \dot{H}^1(E)]_{s,2}$ with $\dot{L}^2(E)$ and $\dot{H}^1(E)$ being the subspaces of the functions of zero average in $L^2(E)$ and $H^1(E)$, respectively. We conclude using Lemma A.2 \square

Lemma A.2. *The spaces $[\dot{L}^2(E), \dot{H}^1(E)]_s$ and $[L^2(E), H^1(E)]_s \cap \dot{L}^2(E)$ are identical and the induced norms are identical, i.e., $\|v\|_{\dot{H}^s(E)} = \|v\|_{H^s(E)}$ for all for all $v \in [L^2(E), H^1(E)]_s \cap \dot{L}^2(E)$.*

Proof. One can use Lemma A1 from Guermond [17] with T being the projection onto $\dot{L}^2(\Omega)$. \square

We now state the main result of this section. It is a variant of Lemma 8.2 in Buffa and Perugia [9] with the extra term $\|\phi\|_{L^2(K)}$. Our proof slightly differs from that in Buffa and Perugia [9] since the proof therein did not appear convincing to us (actually, the embedding inequality at line 9, page 2224 in Buffa and Perugia [9] has a constant that depends on the size of the cell; for instance, using $\phi = 1$ in this inequality yields a contradiction. As result the estimate (8.11) in [9] is not uniform with respect to h).

Lemma A.3. *For all $k \in \mathbb{N}$ and all $\sigma \in (0, \frac{1}{2})$ there is c , uniform with respect to the mesh family, so that the following holds for all faces $F \in \mathcal{F}_h$ in the mesh, all polynomial function \mathbf{v} of degree at most k , and all function $\phi \in \mathbf{H}^\sigma(K) \cap \mathbf{H}(\text{curl}, K)$*

$$(A.3) \quad \int_F (\mathbf{v} \times \mathbf{n}) \cdot \phi \leq c \|\mathbf{v}\|_{L^2(F)} h_F^{-\frac{1}{2}} (h_K^\sigma \|\phi\|_{\mathbf{H}^\sigma(K)} + h_K \|\nabla \times \phi\|_{L^2(K)} + \|\phi\|_{L^2(K)}),$$

where K is either one of the two elements sharing the face F .

Proof. We restrict ourselves to three space dimensions. In two space dimensions ϕ is scalar-valued and the proof must be modified accordingly. Let K be either one of the two elements sharing the face F . Let $\overline{\phi}$ be the average of ϕ over K and let us denote $\psi := \phi - \overline{\phi}$. Upon denoting $\widehat{\mathbf{v}}(\mathbf{x}) = J_K^T \mathbf{v}(T_K(\widehat{\mathbf{x}}))$ and $\widehat{\psi}(\widehat{\mathbf{x}}) = J_K^T \psi(T_K(\widehat{\mathbf{x}}))$, it is a standard result (see Monk [20, 3.82]) that

$$\int_F (\mathbf{v} \times \mathbf{n}) \cdot \psi = \int_{\widehat{F}} (\widehat{\mathbf{v}} \times \widehat{\mathbf{n}}) \cdot \widehat{\psi},$$

where $\widehat{\mathbf{n}}$ is one of the two unit normals on \widehat{F} . Let us extend $\widehat{\mathbf{v}}$ by zero on $\partial\widehat{K} \setminus \widehat{F}$; then $\widehat{\mathbf{v}} \in: \mathbf{H}^{\frac{1}{2}-\sigma}(\partial\widehat{K})$ for all $\sigma > 0$. Note that it is not possible to have $\sigma = 0$. Now let $R : \mathbf{H}^{\frac{1}{2}-\sigma}(\partial\widehat{K}) \rightarrow \mathbf{H}^{1-\sigma}(\widehat{K})$ be a standard lifting operator. There is a constant depending only on \widehat{K} and σ so that

$$\|R\widehat{\mathbf{v}}\|_{L^2(\widehat{K})} + \|\widehat{\mathbf{v}} \times R\widehat{\mathbf{v}}\|_{\mathbf{H}^{-\sigma}(\widehat{K})} \leq c(\widehat{K}, \sigma) \|R\widehat{\mathbf{v}}\|_{\mathbf{H}^{1-\sigma}(\widehat{K})} \leq c' c(\widehat{K}, \sigma) \|\widehat{\mathbf{v}}\|_{\mathbf{H}^{\frac{1}{2}-\sigma}(\widehat{F})},$$

where $\widehat{\nabla} \times$ is the curl operator in the coordinate system of \widehat{K} . Then we have

$$\begin{aligned} \left| \int_{\widehat{F}} (\widehat{\mathbf{v}} \times \widehat{\mathbf{n}}) \cdot \widehat{\boldsymbol{\psi}} \right| &= \left| \int_{\widehat{K}} \left((R\widehat{\mathbf{v}}) \cdot \widehat{\nabla} \times \widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\psi}} \cdot \widehat{\nabla} \times (R\widehat{\mathbf{v}}) \right) \right| \\ &\leq c \left(\|(R\widehat{\mathbf{v}})\|_{\mathbf{L}^2(\widehat{K})} \|\widehat{\nabla} \times \widehat{\boldsymbol{\psi}}\|_{\mathbf{L}^2(\widehat{K})} + \|\widehat{\boldsymbol{\psi}}\|_{\mathbf{H}_0^\sigma(\widehat{K})} \|\widehat{\nabla} \times (R\widehat{\mathbf{v}})\|_{\mathbf{H}^{-\sigma}(\widehat{K})} \right) \\ &\leq c \left(\|\widehat{\nabla} \times \widehat{\boldsymbol{\psi}}\|_{\mathbf{L}^2(\widehat{K})} + \|\widehat{\boldsymbol{\psi}}\|_{\mathbf{H}_0^\sigma(\widehat{K})} \right) \|\widehat{\mathbf{v}}\|_{\mathbf{H}^{\frac{1}{2}-\sigma}(\widehat{F})} \\ &\leq c \left(\|\widehat{\nabla} \times \widehat{\boldsymbol{\psi}}\|_{\mathbf{L}^2(\widehat{K})} + \|\widehat{\boldsymbol{\psi}}\|_{\mathbf{H}^\sigma(\widehat{K})} \right) \|\widehat{\mathbf{v}}\|_{\mathbf{H}^{\frac{1}{2}-\sigma}(\widehat{F})}, \end{aligned}$$

where we used that $\mathbf{H}^\sigma(\widehat{K}) = \mathbf{H}_0^\sigma(\widehat{K})$ for $\sigma \in [0, \frac{1}{2})$. Due to norm equivalence for discrete functions over \widehat{K} and using that $\|J_K\| \leq ch_K$, $h_K/h_F \leq c$ and $|F| \leq ch_F^2$ in three space dimensions, where c depends of the shape-regularity constant of the mesh family and the polynomial degree k , we have

$$\|\widehat{\mathbf{v}}\|_{\mathbf{H}^{\frac{1}{2}-\sigma}(\widehat{F})} \leq c \|\widehat{\mathbf{v}}\|_{\mathbf{L}^2(\widehat{F})} \leq c \|J_K\| |F|^{-\frac{1}{2}} \|\mathbf{v}\|_{\mathbf{L}^2(F)} \leq ch_K h_F^{-1} \|\mathbf{v}\|_{\mathbf{L}^2(F)} \leq c' \|\mathbf{v}\|_{\mathbf{L}^2(F)}.$$

Using the identity (see Monk [20, Cor. 3.58])

$$(\nabla \times \boldsymbol{\psi})(T_K(\widehat{\mathbf{x}})) = \frac{1}{\det(J_K)} J_K (\widehat{\nabla} \times \widehat{\boldsymbol{\psi}})(\widehat{\mathbf{x}}),$$

we obtain

$$\|\widehat{\nabla} \times \widehat{\boldsymbol{\psi}}\|_{\mathbf{L}^2(K)} \leq c |\det(J_K)|^{\frac{1}{2}} \|J_K^{-1}\| \|\nabla \times \boldsymbol{\psi}\|_{\mathbf{L}^2(K)} \leq ch_K^{\frac{1}{2}} \|\nabla \times \boldsymbol{\psi}\|_{\mathbf{L}^2(K)}.$$

Since the average of $\boldsymbol{\psi}$ over K is zero, we can use Lemma A.1 (with an extra scaling by $\|J_K\|$ for $\widehat{\boldsymbol{\psi}} = J_K^T \boldsymbol{\psi}(T_K)$) to deduce

$$\|\widehat{\boldsymbol{\psi}}\|_{\mathbf{H}^\sigma(\widehat{K})} \leq ch_K^{\sigma-\frac{1}{2}} \|\boldsymbol{\psi}\|_{\mathbf{H}^\sigma(K)}.$$

In conclusion we have obtained the following estimate:

$$\int_F (\mathbf{v} \times \mathbf{n}) \cdot (\boldsymbol{\phi} - \overline{\boldsymbol{\phi}}) \leq c (h_K \|\nabla \times \boldsymbol{\phi}\|_{\mathbf{L}^2(K)} + h_K^\sigma \|\boldsymbol{\phi} - \overline{\boldsymbol{\phi}}\|_{\mathbf{H}^\sigma(K)}) h_K^{-\frac{1}{2}} \|\mathbf{v}\|_{\mathbf{L}^2(F)}.$$

Observing that $\|1\|_{\mathbf{H}^\sigma(K)} \leq \|1\|_{\mathbf{L}^2(K)}^{1-\sigma} \|1\|_{\mathbf{H}^1(K)}^\sigma = \|1\|_{\mathbf{L}^2(K)} = |K|^{\frac{1}{2}}$, we infer that

$$\|\boldsymbol{\phi} - \overline{\boldsymbol{\phi}}\|_{\mathbf{H}^\sigma(K)} \leq \|\boldsymbol{\phi}\|_{\mathbf{H}^\sigma(K)} + |\overline{\boldsymbol{\phi}}| |K|^{\frac{1}{2}}$$

The Cauchy-Schwarz inequality yields $|\overline{\boldsymbol{\phi}}| \leq |K|^{-\frac{1}{2}} \|\boldsymbol{\phi}\|_{\mathbf{L}^2(K)}$; as a result,

$$\|\boldsymbol{\phi} - \overline{\boldsymbol{\phi}}\|_{\mathbf{H}^\sigma(K)} \leq \|\boldsymbol{\phi}\|_{\mathbf{H}^\sigma(K)} + \|\boldsymbol{\phi}\|_{\mathbf{L}^2(K)} \leq 2\|\boldsymbol{\phi}\|_{\mathbf{H}^\sigma(K)}.$$

Now we evaluate a bound from above on $\int_F (\mathbf{v} \times \mathbf{n}) \cdot \overline{\boldsymbol{\phi}}$ as follows:

$$\begin{aligned} \left| \int_F (\mathbf{v} \times \mathbf{n}) \cdot \overline{\boldsymbol{\phi}} \right| &\leq |\overline{\boldsymbol{\phi}}| |F|^{\frac{1}{2}} \|\mathbf{v}\|_{\mathbf{L}^2(F)} \leq |K|^{-\frac{1}{2}} \|\boldsymbol{\phi}\|_{\mathbf{L}^2(K)} |F|^{\frac{1}{2}} \|\mathbf{v}\|_{\mathbf{L}^2(F)} \\ &\leq c \|\mathbf{v}\|_{\mathbf{L}^2(F)} h_F^{-\frac{1}{2}} \|\boldsymbol{\phi}\|_{\mathbf{L}^2(K)}. \end{aligned}$$

The result follows by combining all the above estimates. \square

Lemma A.4. *Let $\alpha \in (\frac{1}{2}, 1)$. There is exists a constant $c(\alpha)$ so that*

$$(A.4) \quad \|u\|_{L^2(\Gamma)} \leq c(\alpha) \|u\|_{L^2(\Omega)}^{1-\frac{1}{2\alpha}} \|u\|_{H^\alpha(\Omega)}^{\frac{1}{2\alpha}}, \quad \forall u \in H^\alpha(\Omega).$$

Similarly, for $s \in (0, \frac{1}{2})$, there exists a constant $c(s)$ so that, for $\Theta := \frac{1-2s}{2(1-s)}$,

$$(A.5) \quad \|u\|_{L^2(\Gamma)} \leq c(s) \|u\|_{H^s(\Omega)}^{1-\Theta} \|u\|_{H^1(\Omega)}^{\Theta}, \quad \forall u \in H^1(\Omega).$$

Proof. We start with the standard estimate

$$\|u\|_{L^2(\Gamma)} \leq c \|u\|_{L^2(\Omega)}^{\frac{1}{2}} \|u\|_{H^1(\Omega)}^{\frac{1}{2}}, \quad \forall u \in H^1(\Omega),$$

which allows us to apply Lemma A.5. This implies that the trace operator is a continuous linear mapping from $[L^2(\Omega), H^1(\Omega)]_{\frac{1}{2},1}$ to $L^2(\Gamma)$. Then the re-iteration lemma implies that

$$\begin{aligned} [L^2(\Omega), H^\alpha(\Omega)]_{\frac{1}{2\alpha},1} &= [L^2(\Omega), [L^2(\Omega), H^1(\Omega)]_{\alpha,2}]_{\frac{1}{2\alpha},1} = [L^2(\Omega), H^1(\Omega)]_{\frac{1}{2},1} \\ [H^s(\Omega), H^1(\Omega)]_{\Theta,1} &= [[L^2(\Omega), H^1(\Omega)]_{s,2}, H^1(\Omega)]_{\Theta,1} = [L^2(\Omega), H^1(\Omega)]_{\frac{1}{2},1} \end{aligned}$$

The norms being equivalent, we can eventually write:

$$\begin{aligned} \|u\|_{L^2(\Gamma)} &\leq c \|u\|_{[L^2(\Omega), H^1(\Omega)]_{\frac{1}{2},1}} \leq c(\alpha) \|u\|_{[L^2(\Omega), H^\alpha(\Omega)]_{\frac{1}{2\alpha},1}} \leq c(\alpha) \|u\|_{L^2(\Omega)}^{1-\frac{1}{2\alpha}} \|u\|_{H^\alpha(\Omega)}^{\frac{1}{2\alpha}}, \\ \|u\|_{L^2(\Gamma)} &\leq c \|u\|_{[L^2(\Omega), H^1(\Omega)]_{\frac{1}{2},1}} \leq c(s) \|u\|_{[H^s(\Omega), H^1(\Omega)]_{\Theta,1}} \leq c(s) \|u\|_{H^s(\Omega)}^{1-\Theta} \|u\|_{H^1(\Omega)}^{\Theta}. \end{aligned}$$

This concludes the proof. \square

Lemma A.5 (Lions-Petree). *Let $E_1 \subset E_0$ be two Banach spaces, with continuous embedding. Let L be a linear mapping $E_1 \rightarrow F$ with F another Banach space. For $s \in (0, 1)$, L extends to a linear mapping from $[E_0, E_1]_{s,1}$ to F if and only if there exists $C > 0$ such that*

$$\forall u \in E_1, \quad \|Lu\|_F \leq C \|u\|_{E_0}^{1-s} \|u\|_{E_1}^s.$$

Proof. See Lemma 25.3 in Tartar [22]. \square

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